

# Point Particle with Extrinsic Curvature as a Boundary of a Nambu-Goto String: Classical and Quantum Model

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## Abstract

It is shown how a string living in a higher dimensional space can be approximated as a point particle with squared extrinsic curvature. We consider a generalized Howe-Tucker action for such a “rigid particle” and consider its classical equations of motion and constraints. We find that the algebra of the Dirac brackets between the dynamical variables associated with velocity and acceleration contains the spin tensor. After quantization, the corresponding operators can be represented by the Dirac matrices, projected onto the hypersurface that is orthogonal to the direction of momentum. A condition for the consistency of such a representation is that the states must satisfy the Dirac equation with a suitable effective mass. The Pauli-Lubanski vector composed with such projected Dirac matrices is equal to the Pauli-Lubanski vector composed with the usual, non projected, Dirac matrices, and its eigenvalues thus correspond to spin one half states.

## 1 Introduction

Extended objects, such as branes with extrinsic curvature are of great interest for physics [1]–[9]. A particular case is the point particle with extrinsic curvature, the so called “rigid particle” [10]–[23]. Such an object, because of the second derivatives in the action, moves along a trajectory that is not a straight line, but a helix. The rectilinear component of the helical worldline corresponds to the particle’s momentum  $p_\mu$ , whereas the circular component is responsible for spin,  $S_{\mu\nu}$ . The quantities  $p_\mu$ ,  $S_{\mu\nu}$  and the orbital momentum  $L_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu$  satisfy the relations of a classical particle with spin [24]–[27]. A question arises as to whether the rigid particle can be a classical model for the quantum particle with spin, described by the Dirac equation. In fact, there are two types of rigid particles: those with the extrinsic curvature to the power one (type 1) [5]–[17], and those with the squared intrinsic curvature (type 2) [18]–[23], [11].

It was shown [11] that if one starts from an ordinary Nambu-Goto string (without extrinsic curvature) living in a space with one extra space-like dimension, then one can derive type 2 rigid particle as an approximation. According to the authors of Ref. [11], such derivation was not quite consistent. In Ref. [21], it was shown how we can obtain the consistent rigid particle: the squared extrinsic curvature with the correct sign in the rigid particle’s action comes from a string living in a spacetime with an extra time-like dimension. Starting from the Nambu-Goto string action, one can directly arrive at the type 2 rigid particle ac-

tion [11, 21]. In this paper I will show, following the previous work [21], that if we start from the Polyakov form of the string action, then as an intermediate approximation we obtain a *generalized Howe-Tucker action* that contains second order derivative<sup>1</sup>. We will study the classical and quantum equations of such a generalized Howe-Tucker point particle action, describing what we will call type 2a rigid particle. The system contains two first class and four second class constraints [28]. The Dirac brackets between the phase space variables associated with velocity contains the spin tensor. The similar holds for the Dirac brackets associated with acceleration. In the quantized theory, those Dirac bracket relations become the commutation relations between the operators [28]. It turns out that these operators can be represented in terms of the gamma matrices, multiplied by the generators of the Clifford algebra  $Cl(0, 2)$  of a 2-dimensional space with signature  $(--)$ . The latter space is a subspace of the phase space of our dynamical system. The signature  $(--)$  comes from the space like type of the chosen dynamical variables, the 4-acceleration and the projection of the 4-velocity onto a space like hypersurface. The Pauli-Lubanski vector turns out to be the same as that for a Dirac particle. The analysis presented in this paper thus leads to a conclusion that the type 2a rigid particle, upon quantization, has spin  $1/2$ . A. Deriglazov [29] considered type 1 rigid particle, whose dynamics is different, but he also found that the phase space variables can be quantized by gamma matrices and that the system has spin one-half.

The physical states must satisfy the conditions imposed by the first class constraints. This can be consistent with the second class constraints and the representation of the operators in terms of the Clifford numbers if we bring an additional time-like dimension into the game, besides the two ones considered so far in our model.

In Sec. 2 we describe a scenario with an open string living in a  $(D+1)$ -dimensional target space whose  $(D+1)^{\text{th}}$  dimension, as well as the  $1^{\text{th}}$  one, are time-like. For the  $(D+1)^{\text{th}}$  embedding function we choose  $X^{D+1}(\tau, \sigma) = \sigma$  which in the considered scenario, illustrated in Fig. 1, is possible because of the reparametrization invariance of the string action. We expand the string embedding functions  $X^\mu(\tau, \sigma)$ ,  $\mu = 0, 1, 2, \dots, D$ . into the Taylor series around  $\sigma = 0$ . If we take only two terms of the latter expansions, then we obtain an action for a point particle which includes second order derivatives. In Sec. 3 we consider the dynamics of such a particle. We compute the Hamiltonian, the corresponding equations of motion, the first and the second class constraints and the Dirac brackets between certain dynamical variables. In Sec. 4 we perform quantization of our system in the Schrödinger and in the Heisenberg picture. In Conclusion we summarize our results and point out why they are important for further progress on our road to the unified theory of fields and particles, including quantum gravity. In Appendix we consider a particular solution to the Laplace equation satisfied by the time-like string, according to which the string end at  $\sigma = 0$  moves as a point particle with extrinsic curvature.

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<sup>1</sup> If in the latter action we perform a further approximation, then we obtain the type 2 rigid particle action.

## 2 The string with time-like extension

Let us consider an open string, embedded in a  $(D + 1)$ -dimensional target space,  $M_{D+1}$ , such that the string ends are attached to two  $Dp$ -branes [30], with  $p = D - 1$ , that sweep the worldvolumes  $V_D$  and  $V'_D$ , as shown in Fig.1. The extra,  $(D + 1)^{\text{th}}$  dimension need not be compact, it can extend to infinity. If so, we can adopt the braneworld scenario (see, e.g., a review [32]), and assume that our world is in one of the two worldvolumes, say, in  $V_D$ . Since we observe the 4-dimensional spacetime, we may take  $D = 4$ , or assume that  $D - 4$  dimensions of  $V_D$  are compactified. Alternatively, we can assume that  $V_D$  is the 16-dimensional Clifford space [31], i.e., the space of points, areas, volumes and 4-volumes associated with physical objects living in 4-dimensional spacetime. We are interested in how the string end moves in  $V_D$ . From the point of view of an observer in  $V_D$ , the motion of the string end is perceived as the motion of a point particle.

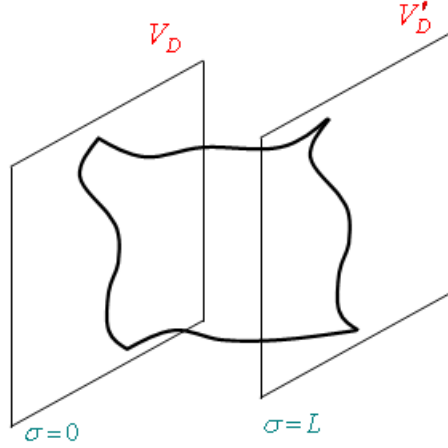


Figure 1: A scenario in which a string's worldsheet is bounded by two worldvolumes  $V_D$  and  $V'_D$ , corresponding to two  $Dp$ -branes with  $p = D - 1$ .

The string is described by the Nambu-Goto action in  $(D + 1)$ -dimensions:

$$I = T \int d^2\xi (\epsilon \hat{f})^{1/2}, \quad (1)$$

where  $T$  is the string tension, and  $\hat{f} = \det \hat{f}_{ab}$ ,  $\hat{f}_{ab} = \partial_a X^{\hat{\mu}} \partial_b X_{\hat{\mu}}$ ,  $\hat{\mu} = (\mu, D + 1)$ ,  $\mu = 0, 1, 2, \dots, D - 1$ ,  $\xi^a = (\xi^1, \xi^2) \equiv (\tau, \sigma)$ . Here  $\epsilon = \pm 1$  depends on the signature of the metric on the worldsheet  $V_2$  swept by the string:  $\epsilon = -1$  corresponds to the signature  $(+ -)$ , whereas  $\epsilon = 1$  corresponds to the signature  $(++)$ . Because the action (1) is invariant under reparametrizations of  $\tau$  and  $\sigma$ , we have a certain freedom in the choice of the string embedding functions  $X^{\hat{\mu}}(\tau, \sigma)$ . In the scenario, illustrated in Fig. 1, this enables us to set

$$X^{D+1}(\tau, \sigma) = \sigma \quad (2)$$

Let us now expand the remaining embedding functions  $X^\mu(\tau, \sigma)$ ,  $\mu = 0, 1, 2, \dots, D - 1$ ,

into the Taylor series around  $\sigma = 0$ :

$$X^\mu(\tau, \sigma) = X^\mu(\tau, 0) + \left. \frac{\partial X^\mu}{\partial \sigma} \right|_0 \sigma + \frac{1}{2} \left. \frac{\partial^2 X^\mu}{\partial \sigma^2} \right|_0 \sigma^2 + \dots \quad (3)$$

Introducing

$$x^\mu(\tau) \equiv X^\mu(\tau, 0), \quad y_1^\mu(\tau) \equiv \left. \frac{1}{k} \frac{\partial X^\mu}{\partial \sigma} \right|_0, \quad y_2^\mu(\tau) \equiv \left. \frac{1}{k^2} \frac{\partial^2 X^\mu}{\partial \sigma^2} \right|_0 \quad (4)$$

the expansion (3) reads

$$X^\mu(\tau, \sigma) = x^\mu(\tau) + y_1^\mu(\tau) k \sigma + \frac{1}{2} y_2^\mu(\tau) k^2 \sigma^2 + \dots, \quad (5)$$

where  $k$  is a constant, such that the product  $k\sigma$  is dimensionless. We see that the string can be described in terms of infinite number of the  $\tau$ -dependent functions  $x^\mu(\tau)$ ,  $y_i^\mu(\tau)$ ,  $i = 1, 2, \dots, \infty$ . If the string is short, so that  $kL \ll 1$ , already first few terms in the expansion (5) will provide, within a prescribed accuracy, a sufficiently good description of the string, satisfying exactly the boundary conditions on one end, and approximately on the other end.

The functions  $X^\mu(\tau, \sigma)$ , expanded according to Eq. (5), can be inserted into the action (1). After performing the integration over  $\sigma$  from 0 to  $L$ , we obtain an action functional for an infinite set of  $\tau$ -dependent variables  $x^\mu(\tau)$ ,  $y_i^\mu(\tau)$ ,  $i = 1, 2, \dots, \infty$ . If we take a finite number of terms in the expansion (5), then we obtain an action which is a functional of a finite number of variables  $x^\mu(\tau)$ ,  $y_i^\mu(\tau)$ ,  $i = 1, 2, \dots, n$ . Our string with infinitely many degrees of freedom is thus sampled by a finite number of degrees of freedom. Descriptions of physical objects in terms of an effective action are common in physics. Instead of taking into account all the degrees of freedom, we can sample the object by less degrees of freedom. For instance, though the Earth is an object with very many degrees of freedom, we can describe its motion around the Sun, by taking into account only its center of mass degrees of freedom. We do not care about other variables and boundary conditions determining their motion. Similarly, effective actions in which certain degrees of freedom of a system have been integrated out, are commonly used in high energy physics. According to (5), a string can also be described by an effective action, and the corresponding “effective” equations of motion for a finite set of the variables  $x^\mu(\tau)$ ,  $y_i^\mu(\tau)$ . Then, instead of the boundary and initial conditions for the variables  $X^\mu(\tau, \sigma)$ , we have to specify the initial conditions for  $x^\mu(\tau)$ ,  $y_i^\mu(\tau)$  only. Since we are interested only in the behavior of a finite number of the variables  $x^\mu(\tau)$ ,  $y_i^\mu(\tau)$ , describing the motion of the string end at  $\sigma = 0$ , and not in the behavior of the entire string, we do not need to take into account the boundary conditions for the string end at  $\sigma = L$ .

Let us now denote  $y^\mu(\tau) \equiv y_1^\mu(\tau)$ , write the expansion (5) as

$$\begin{aligned} X^\mu(\tau, \sigma) &= x^\mu(\tau) + y^\mu(\tau) k \sigma + \dots \\ X^{D+1}(\sigma) &= \sigma, \end{aligned} \quad (6)$$

and neglect all higher order terms. Then, up to such approximation, the induced metric is

$$f_{ab} = \partial_a X^{\hat{\mu}} \partial_b X_{\hat{\mu}} = \begin{pmatrix} \dot{x}^2 + 2\dot{x}y k \sigma, & \dot{x}y k \\ \dot{x}y k, & k^2 y^2 + \epsilon \end{pmatrix} + \dots, \quad (7)$$

and the string action (1) becomes [11, 21]

$$I = T \int_0^L d\tau d\sigma \sqrt{\dot{x}^2} \left( 1 + \frac{\epsilon}{2} k^2 y^2 + \frac{\dot{x}y k \sigma}{\dot{x}^2} - \frac{\epsilon k^2 (\dot{x}y)^2}{2\dot{x}^2} \right) + \mathcal{O}(k^2 L^2), \quad (8)$$

which, after the integration over  $\sigma$ , becomes

$$I = TL \int d\tau \sqrt{\dot{x}^2} \left( 1 + \frac{\epsilon}{2} k^2 y^2 + \frac{\dot{x}y k L}{2\dot{x}^2} - \frac{\epsilon k^2 (\dot{x}y)^2}{2\dot{x}^2} \right) + \mathcal{O}(k^2 L^2), \quad (9)$$

Variation of the latter action with respect to  $y^\mu$  gives

$$y^\mu = -\frac{L}{2k} H^\mu + \frac{1}{\dot{x}^2} (\dot{x}^\alpha y_\alpha) \dot{x}^\mu, \quad H^\mu \equiv \frac{1}{\sqrt{\dot{x}^2}} \frac{d}{d\tau} \left( \frac{\dot{x}^\mu}{\sqrt{\dot{x}^2}} \right), \quad (10)$$

where  $H^\mu \dot{x}_\mu = 0$ . By inserting the expression (10) into the action (8), and by writing  $m = TL$  and  $\mu = TL^3/8$ , we obtain

$$I = \int d\tau \sqrt{\dot{x}^2} (m - \epsilon \mu H^2) + \mathcal{O}(k^2 L^2). \quad (11)$$

This is the action for the type 2 rigid particle, if  $\epsilon = 1$ , i.e., if the worldsheet signature is  $(++)$  and the  $(D+1)$ -th dimension is time-like.

Instead of the Nambu-Goto action (1), we can consider the Polyakov action

$$I[X^{\hat{\mu}}, \gamma^{ab}] = \frac{T}{2} \int d^2\xi \sqrt{\epsilon\gamma} \gamma^{ab} \partial_a X^{\hat{\mu}} \partial_b X_{\hat{\mu}}, \quad (12)$$

which gives the same classical equations of motion.

From now on, we will consider the case where  $\epsilon = 1$ . Then the signature of the target space is  $(2, D-1)$ , which means two time-like and  $D-1$  space-like dimensions. The direction along  $\sigma$  is bounded, whereas the direction along  $\tau$  is open. Until the seminal works by Bars et al. [33], it was generally believed that in the presence of extra time-like dimensions there must necessarily be ghosts that cannot be eliminated, a consequence being that such theories are inconsistent. However, the so called 2-time (2T) physics, developed in Refs. [33], is quite consistent and gives numerous remarkable results. Moreover, in Refs. [34]–[40] it was shown that also the theories in ultra hyperbolic spaces, including the self-interacting Pais-Uhlenbeck oscillators, can be consistent and stable. Because of those encouraging results, it makes sense to consider as well the target space with the signature  $(2, D-1)$  in which there lives a time-like string, sweeping a worldsheet with the signature  $(++)$ .

By taking the expansion (6) and inserting the metric (7) into the action (12), we obtain

$$I = \frac{T}{2} \int d\tau d\sigma \sqrt{\gamma} [\gamma^{11} (\dot{x}^2 + 2\dot{x}y k \sigma) + 2\gamma^{12} k \dot{x}y + \gamma^{22} (k^2 y^2 + 1)] + \mathcal{O}(k^2 L^2). \quad (13)$$

We now use  $\gamma^{ab} = (1/\gamma)\partial\gamma/\partial\gamma_{ab}$ , and write

$$\sqrt{\gamma}\gamma^{11} = \frac{1}{\sqrt{\gamma}}\gamma_{22} = \frac{1}{E(\tau, \sigma)} = \frac{1}{e(\tau)} + \frac{\partial E^{-1}}{\partial \sigma} \Big|_0 \sigma + \dots \quad (14)$$

$$-\sqrt{\gamma}\gamma^{12} = \frac{1}{\sqrt{\gamma}}\gamma_{12} = F(\tau, \sigma) = f(\tau) + \frac{\partial F}{\partial \sigma} \Big|_0 \sigma + \dots \quad (15)$$

$$\sqrt{\gamma}\gamma^{22} = \frac{1}{\sqrt{\gamma}}\gamma_{11} = \frac{1}{E(\tau, \sigma)} = E(1 + F^2) = e(1 + f^2) + \mathcal{O}(\sigma) \quad (16)$$

Here  $\gamma = \gamma_{11}\gamma_{22} - \gamma_{12}^2 = \gamma[E(1 + F^2)\frac{1}{E} - F^2]$ , which justifies the above parametrization.

If we insert Eqs. (14)–(16) into the action (13) and integrate over  $\sigma$ , we obtain

$$I = \frac{LT}{2} \int d\tau \left[ \frac{1}{e}(\dot{x}^2 + Lk\dot{x}\dot{y}) + e(1 + f^2)(k^2y^2 + 1) - 2fk\dot{x}\dot{y} \right] + \mathcal{O}(k^2L^2). \quad (17)$$

The equations of motion are:

$$\delta e : \quad -\frac{1}{e^2}(\dot{x}^2 + Lk\dot{x}\dot{y}) + (1 + f^2)(1 + k^2y^2) = 0, \quad (18)$$

$$\delta f : \quad fe(1 + k^2y^2) - k\dot{x}\dot{y} = 0, \quad (19)$$

$$\delta y : -Lk \frac{d}{d\tau} \left( \frac{\dot{x}^\mu}{e} \right) + 2e(1 + f^2)y^\mu - 2fk\dot{x}^\mu = 0, \quad (20)$$

$$\delta x : \quad \frac{d}{d\tau} \left( \frac{\dot{x}^\mu}{e} + \frac{Lk\dot{y}^\mu}{2e} - fky^\mu \right) = 0. \quad (21)$$

Here  $e$  and  $f$  are the Lagrange multipliers. Choice of a Lagrange multiplier means choice of gauge. Recall that, according to Eqs. (14)–(16),  $e$  and  $f$  are related to the string metric. The string action is invariant under reparametrizations of  $\tau$  and  $\sigma$ , a consequence being the existence of two constraints. By a judicious choice of parameters  $\tau$  and  $\sigma$ , one can transform the string metric into a diagonal form (though not necessarily into the conformal one), so that  $\gamma^{12}$  vanish. But vanishing of  $\gamma^{12}$  means vanishing of  $f$ .

If we choose  $f = 0$ , then Eqs. (19),(20) give:

$$y^\mu \dot{x}_\mu = 0, \quad (22)$$

$$y^\mu = \frac{L}{2ke} \frac{d}{d\tau} \left( \frac{\dot{x}^\mu}{e} \right) = \frac{L}{2k} \left( \frac{\dot{x}^2}{e^2} H^\mu + \frac{1}{e} \frac{\dot{x}^\mu}{\sqrt{\dot{x}^2}} \frac{d}{d\tau} \left( \frac{\sqrt{\dot{x}^2}}{e} \right) \right), \quad (23)$$

where

$$H^\mu \equiv \frac{1}{\sqrt{\dot{x}^2}} \frac{d}{d\tau} \left( \frac{\dot{x}^\mu}{\sqrt{\dot{x}^2}} \right). \quad (24)$$

From Eqs. (22)–(24) we find

$$\frac{d}{d\tau} \left( \frac{\dot{x}^\mu}{e} \right) \frac{\dot{x}^\mu}{\sqrt{\dot{x}^2}} = H^\mu \dot{x}_\mu \frac{\sqrt{\dot{x}^2}}{e} + \frac{d}{d\tau} \left( \frac{\sqrt{\dot{x}^2}}{e} \right) = 0. \quad (25)$$

By using the identity  $H^\mu \dot{x}_\mu = 0$ , we then obtain:

$$\frac{d}{d\tau} \left( \frac{\sqrt{\dot{x}^2}}{e} \right) = 0, \quad (26)$$

$$y^\mu = \frac{L}{2k} \frac{\dot{x}^2}{e^2} H^\mu, \quad (27)$$

$$y^\mu y_\mu \equiv y^2 = \left( \frac{L}{2k} \right)^2 \left( \frac{\dot{x}^2}{e^2} \right)^2 H^2. \quad (28)$$

We see that  $y^2$  is proportional to the squared extrinsic curvature  $H^2 \equiv H^\mu H_\mu$ .

With  $f = 0$ , the constraint (18) reads

$$-\frac{1}{e^2}(\dot{x}^2 + Lk\dot{x}\dot{y}) + 1 + k^2 y^2. \quad (29)$$

From Eqs. (22),(23) it is not difficult to find

$$\dot{x}\dot{y} = -\frac{2k}{L} e^2 y^2. \quad (30)$$

Inserting this into Eq. (29), we obtain:

$$\frac{\dot{x}^2}{e^2} = 1 + 3k^2 y^2. \quad (31)$$

Because  $\dot{x}^2/e^2$  is a constant of motion (c.f. Eq.(26)), also  $y^2$  and  $H^2$  are constants of motion.

From Eqs. (27) and (31) we have

$$\frac{\dot{x}^2}{e^2} = 1 + \frac{6\mu}{m} \left( \frac{\dot{x}^2}{e^2} \right)^2 H^2, \quad (32)$$

where we have introduced

$$m = LT \quad \text{and} \quad \mu = \frac{L^3 T}{8}. \quad (33)$$

The solution of Eq. (32) is

$$\frac{\dot{x}^2}{e^2} = \frac{1 \pm \sqrt{1 - \frac{24\mu}{m} H^2}}{\frac{12\mu H^2}{m}}. \quad (34)$$

Assuming that  $24\mu H^2/m \ll 1$ , we obtain:

$$\text{I.} \quad \frac{\dot{x}^2}{e^2} \approx \frac{m}{6\mu H^2}, \quad (35)$$

$$\text{II.} \quad \frac{\dot{x}^2}{e^2} \approx 1 + \frac{6\mu H^2}{m}. \quad (36)$$

Solution I is inconsistent with our assumption that  $\dot{x}^2 > 0$  and  $H^2 < 0$ . Therefore we take Solution II, and write

$$\frac{\sqrt{\dot{x}^2}}{e} \approx \sqrt{1 + \frac{6\mu H^2}{m}} \approx 1 + \frac{3\mu H^2}{m}. \quad (37)$$

By using Eqs. (30),(37),(33) and  $f = 0$ , the action (17) becomes

$$I = \int d\tau \sqrt{\dot{x}^2} (m - \mu H^2) + \mathcal{O} \left( m \left( \frac{\mu H^2}{m} \right)^2 \right), \quad (38)$$

which is the *rigid particle action*.

We see that as a first approximation to the string action (12) we obtain the point particle action (17). A further approximation is in expanding  $\dot{x}^2/e^2$  according to (36) and then  $\sqrt{\dot{x}^2}/e$  according to (37). Then we obtain the type 2 rigid particle action, apart from the term  $m(\mu H^2/m)^2$  and the higher order terms that we neglect. In the rest of the paper we will consider the “intermediate”, type 2a, action (17), and its equivalent forms.

### 3 The dynamics of the spinning point particle derived from the string

In the previous section we derived from the Polyakov string action (12) an effective point particle action which for  $f = 0$  reads

$$I = \frac{LT}{2} \int d\tau \left[ \frac{\dot{x}^2}{e} + e + \frac{Lk\dot{x}\dot{y}}{e} + ek^2y^2 \right]. \quad (39)$$

Let me clarify again that this action is an approximation to the Polyakov action in the sense that it does not describe all the degrees of freedom of the string; it describes the motion of the string end at  $\sigma = 0$ , attached to the  $Dp$ -brane  $V_{D-1}$ . Even less degrees of freedom we take into account if we consider the expression  $X^\mu(\tau, \sigma) = x^\mu(\tau)$ , in which we neglect all terms in the expansion (6), except the first one, i.e., if we take  $k = 0$ . Then, instead of the action (39), we obtain the well known Howe-Tucker [41] action for a point particle: it describes the center of mass motion of the string. The boundary conditions for the string’s ends are irrelevant for its center of mass. Instead of the string equations of motion we have then the equations of motion for its center of mass. What we have to specify in such a case are the initial conditions for the center of mass coordinates. The action (39), besides  $x^\mu(\tau)$ , contains also the  $y^\mu(\tau)$  degrees of freedom that are related to the extrinsic curvature, making the twist along a helix, as illustrated in Fig. 2 (see Appendix). The motion of the string end at  $\sigma = L$  is irrelevant for the degrees of freedom  $x(\tau)$  and  $y(\tau)$ , describing, respectively, the motion of the string’s end at  $\sigma = 0$ , and the twist along a helix. Notice that the parameter  $\sigma$  does not occur at all in the degrees of freedom  $x$  and  $y$ . Having in mind this fact, it is obvious, why in the quenched description of the string in terms of  $x$  and  $y$  it is not necessary to take into account the string’s boundary conditions. Boundary



conditions are relevant for those degrees of freedom that depend on  $\sigma$ . Here, the degrees of freedom  $x^\mu$  and  $y^\mu$  depend on  $\tau$  only, therefore their evolution in  $\tau$  can be determined from the equations of motion if one provides initial conditions, e.g., at  $\tau = 0$ .

If we plug the solution

$$y^\mu = \frac{L}{2k} \frac{1}{e} \frac{d}{d\tau} \left( \frac{\dot{x}^\mu}{e} \right) \quad (40)$$

into the action (39) and introduce parameters  $m$  and  $\mu$  according to (33), then we obtain<sup>2</sup>

$$I[x^\mu, e] = \int d\tau \left[ \frac{m}{2} \left( \frac{\dot{x}^2}{e} + e \right) - \frac{\mu}{e} \frac{d}{d\tau} \left( \frac{\dot{x}^\mu}{e} \right) \frac{d}{d\tau} \left( \frac{\dot{x}_\mu}{e} \right) \right]. \quad (41)$$

The action (41) contains second order derivative. We will now employ the Ostrogradsky formalism [42] and transform (41) into the Hamilton form. The momenta are

$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} - \frac{d}{d\tau} \left( \frac{\partial L}{\partial \ddot{x}^\mu} \right) = \frac{m \dot{x}_\mu}{e} + \frac{2\mu}{e} \frac{d}{d\tau} \left( \frac{1}{e} \frac{d}{d\tau} \left( \frac{\dot{x}_\mu}{e} \right) \right), \quad (42)$$

$$\pi_\mu = \frac{\partial L}{\partial \ddot{x}^\mu} = -\frac{2\mu}{e^2} \frac{d}{d\tau} \left( \frac{\dot{x}_\mu}{e} \right), \quad (43)$$

$$p_e = \frac{\partial L}{\partial \dot{e}} = \frac{2\mu}{e^3} \dot{x}^\mu \frac{d}{d\tau} \left( \frac{\dot{x}_\mu}{e} \right). \quad (44)$$

The equation of motion for  $x^\mu$  is:

$$\delta x^\mu : \quad \dot{p}_\mu = 0 \quad . \quad (45)$$

The equation of motion for  $e$  is

$$\delta e : \quad \frac{\partial L}{\partial e} - \frac{d}{d\tau} \frac{\partial L}{\partial \dot{e}} = 0, \quad (46)$$

which gives

$$\frac{m}{2} \left( 1 - \frac{\dot{x}^2}{e^2} \right) + 3\mu \frac{1}{e} \frac{d}{d\tau} \left( \frac{\dot{x}^\mu}{e} \right) \frac{1}{e} \frac{d}{d\tau} \left( \frac{\dot{x}_\mu}{e} \right) - \frac{2\mu}{e} \frac{d}{d\tau} \left( \frac{\dot{x}^\mu}{e^2} \frac{d}{d\tau} \left( \frac{\dot{x}_\mu}{e} \right) \right) = 0. \quad (47)$$

The Hamiltonian is

$$H_0 = p_\mu \dot{x}^\mu + \pi_\mu \ddot{x}^\mu + p_e \dot{e} - L_0, \quad (48)$$

where  $L_0$  is the Lagrangian corresponding to the action (41). By introducing

$$\dot{x}^\mu = q^\mu, \quad \dot{e} = \beta, \quad (49)$$

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<sup>2</sup> Introducing a new parameter  $\tau'$  according to  $d\tau' m \tilde{e} = d\tau e$ , the action (41) assumes the form  $I[x^\mu, \tilde{e}] = \int d\tau' \left[ \frac{1}{2} \left( \frac{\dot{x}^2}{\tilde{e}} + \tilde{e} m^2 \right) - \frac{\mu}{m^3 \tilde{e}} \frac{d}{d\tau'} \left( \frac{\dot{x}^\mu}{\tilde{e}} \right) \frac{d}{d\tau'} \left( \frac{\dot{x}_\mu}{\tilde{e}} \right) \right]$ , where we have renamed  $\tau'$  into  $\tau$ . From Eq. (33) we have  $\mu/m^3 = 1/(8T^2) \equiv \tilde{\mu}$ . In the limit  $m \rightarrow 0$ , such that  $L \rightarrow 0$ ,  $T = finite$ , the latter action becomes identical to the action for the "massless" particle with curvature, considered by McKeon [19].

and by inserting (see Eqs. (43) and (44))

$$\ddot{x}^\mu = -\frac{e^3}{2\mu}\pi^\mu + \frac{\dot{e}}{e}q^\mu, \quad (50)$$

$$p_e = -\frac{\pi_\mu q^\mu}{e} \quad (51)$$

into Eq. (48), we obtain

$$H_0 = p_\mu q^\mu - \frac{e^3 \pi^2}{4\mu} - \frac{m}{2} \left( \frac{q^2}{e} + e \right) + \beta(p_e + \frac{\pi_\mu q^\mu}{e}), \quad (52)$$

and

$$L_0 = p_\mu \dot{x}^\mu + \pi_\mu \dot{q}^\mu + p_e \dot{e} - \left[ p_\mu q^\mu - \frac{e^3 \pi^2}{4\mu} - \frac{m}{2} \left( \frac{q^2}{e} + e \right) \right] - \beta(p_e + \frac{\pi_\mu q^\mu}{e}). \quad (53)$$

This Lagrangian gives the equations of motion which are equivalent to Eqs. (42)–(46).

Let us now take into account the fact that the action (39) holds for  $f = 0$  which, according to Eq. (19), implies

$$\dot{x}^\mu y_\mu = 0. \quad (54)$$

In view of Eqs. (40), (43) and (49), the above relation reads

$$\pi_\mu q^\mu = 0. \quad (55)$$

By imposing the latter constraint, the Lagrangian (53) is extended to

$$L = p_\mu \dot{x}^\mu + \pi_\mu \dot{q}^\mu + p_e \dot{e} - \left[ p_\mu q^\mu - \frac{e^3 \pi^2}{4\mu} - \frac{m}{2} \left( \frac{q^2}{e} + e \right) \right] - \alpha \pi_\mu q^\mu - \beta(p_e + \frac{\pi_\mu q^\mu}{e}). \quad (56)$$

The corresponding Hamiltonian is

$$H = p_\mu q^\mu - \frac{e^3 \pi^2}{4\mu} - \frac{m}{2} \left( \frac{q^2}{e} + e \right) + \alpha \pi_\mu q^\mu + \beta(p_e + \frac{\pi_\mu q^\mu}{e}). \quad (57)$$

It gives the following equations of motion:

$$\dot{x}^\mu = \{x^\mu, H\} = q^\mu, \quad (58)$$

$$\dot{e} = \{e, H\} = \beta, \quad (59)$$

$$\dot{q}^\mu = \{q^\mu, H\} = -\frac{e^3 \pi^\mu}{2\mu} + \alpha q^\mu + \frac{\beta q^\mu}{e}, \quad (60)$$

$$\dot{p}_\mu = \{p_\mu, H\} = 0, \quad (61)$$

$$\dot{\pi}_\mu = \{\pi_\mu, H\} = -\left(p_\mu - \frac{m q_\mu}{e} + \alpha \pi_\mu + \frac{\beta \pi_\mu}{e}\right), \quad (62)$$

$$\dot{p}_e = \{p_e, H\} = -\frac{3e^2 \pi^2}{4\mu} - \frac{m}{2} \left(1 - \frac{q^2}{e^2}\right) - \beta \frac{\pi_\mu q^\mu}{e^2}, \quad (63)$$

Using (51), (55), and (63), we find that our system, described by the Hamiltonian (57), has the following constraints:<sup>3</sup>

$$\tilde{\phi}_1 = \frac{3e^2\pi^2}{4\mu} + \frac{m}{2} \left(1 - \frac{q^2}{e^2}\right) = 0, \quad (64)$$

$$\boxed{\phi_2 = \pi_\mu q^\mu = 0}, \quad (65)$$

where  $q^2 \equiv q^\mu q_\mu$  and  $\pi^2 \equiv \pi^\mu \pi_\mu$ .

From the requirement that those constraints must be conserved, we obtain further constraints:

$$\dot{\phi}_1 = 0 \quad \Rightarrow \quad -\frac{3e^2}{2\mu}(\pi_\mu p^\mu + \alpha\pi^2) - \alpha\frac{mq^2}{e^2} = 0, \quad (66)$$

$$\dot{\phi}_2 = 0 \quad \Rightarrow \quad \boxed{\phi_4 = \frac{p_\mu q^\mu}{e} + \frac{e^2\pi^2}{2\mu} - \frac{mq^2}{e^2} = 0}, \quad (67)$$

$$\dot{\phi}_4 = 0 \quad \Rightarrow \quad -\frac{3e^2 p_\mu \pi^\mu}{2\mu} + \alpha \left( \frac{p_\mu q^\mu}{e} - \frac{2mq^2}{e^2} - \frac{e^2\pi^2}{\mu} \right) = 0. \quad (68)$$

If we subtract Eq. (66) from Eq. (68), we obtain  $\alpha\phi_4 = 0$ . Let us introduce the linear combination

$$\boxed{\phi_1 = -\tilde{\phi}_1 + \phi_4 = \frac{p_\mu q^\mu}{e} - \frac{m}{2} \left(1 + \frac{q^2}{e^2}\right) - \frac{e^2\pi^2}{4\mu}}. \quad (69)$$

The conservation of  $\phi_1$  gives

$$\begin{aligned} \dot{\phi}_1 = 0 \quad &\Rightarrow \quad -\frac{e^2}{\mu} p_\mu \pi^\mu + \alpha\phi_4 = 0, \\ \text{i.e.} \quad &\boxed{\phi_3 = ep_\mu \pi^\mu = 0}. \end{aligned} \quad (70)$$

From Eqs. (70), (64) and (66) it follows that  $\alpha = 0$ . The other Lagrange multiplier,  $\beta$ , has already been determined by Eq. (59).

Further we have

$$\dot{\phi}_3 = 0 \quad \Rightarrow \quad \boxed{\phi_5 = p^2 - \frac{mp_\mu q^\mu}{e} = 0}, \quad (71)$$

$$\dot{\phi}_5 = 0 \quad \Rightarrow \quad \frac{me^2}{2\mu} p_\mu \pi^\mu - \alpha \frac{mp_\mu q^\mu}{e} = 0. \quad (72)$$

Because  $\alpha = 0$ , the last equation gives  $\phi_3 = 0$ , which is not a new constraint.

Finally, from Eq. (51) we obtain the constraint

$$\boxed{\phi_6 = ep_e + \pi_\mu q^\mu = 0}. \quad (73)$$

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<sup>3</sup>The above equations and the analogous equations throughout this paper are valid on the constraint surface  $\Sigma$ , but for simplicity we write the equality symbol “=” instead of a weak equality symbol such as “ $\equiv$ ”.

The conservation of the latter constraint, i.e.,  $\dot{\phi}_6 = \{\phi_6, H\} = 0$ , does not give a new constraint.

We have thus the six constraints  $\phi_\alpha$ ,  $\alpha = 1, 2, 3, 4, 5, 6$ . The reason why we have introduced the linear combination  $\phi_1 = -\tilde{\phi}_1 + \phi_4$  (see Eq. (69)), is in the fact that its Poisson brackets with all the remaining constraints vanish. The same holds for  $\phi_6$ . Therefore,  $\phi_1$  and  $\phi_6$  are *first class constraints*, whereas  $\phi_{\bar{\alpha}}$ ,  $\bar{\alpha} = 2, 3, 4, 5$ , are *second class constraints*.

In summary, we have *two first class constraints*

$$\phi_1 = \frac{p_\mu q^\mu}{e} - \frac{m}{2} \left( 1 + \frac{q^2}{e^2} \right) - \frac{e^2 \pi^2}{4\mu} \quad (74)$$

$$\phi_6 = ep_e + \pi_\mu q^\mu \quad (75)$$

and *four second class constraints*

$$\phi_2 = \pi_\mu q^\mu \quad (76)$$

$$\phi_3 = ep_\mu \pi^\mu \quad (77)$$

$$\phi_4 = \frac{p_\mu q^\mu}{e} + \frac{e^2 \pi^2}{2\mu} - \frac{mq^2}{e^2} \quad (78)$$

$$\phi_5 = p^2 - \frac{mp_\mu q^\mu}{e}. \quad (79)$$

The presence of two first class constraints is a result of the fact that we derived our rigid particle from the string, which has two gauge degrees of freedom, related to the worldsheet parameters  $\tau$  and  $\sigma$ . A reparametrization of those two parameters induces a change of  $x^\mu$  and  $y^\mu$  in the expansion (6), which is reflected in a change of our dynamical variables, such a change being generated by the first class constraints  $\phi_1$  and  $\phi_6$ .

If we write the total Hamiltonian,  $H_{\text{tot}} = H + \lambda_3 \phi_3 + \lambda_4 \phi_4 + \lambda_5 \phi_5$ , where  $H$  is given in Eq. (57), and calculate

$$\dot{\phi}_i = \{\phi_i, H_{\text{tot}}\}, \quad i = 3, 4, 5 \quad (80)$$

we find  $\lambda_3 = \lambda_4 = \lambda_5 = 0$ . According to the definition (57),  $H$  is a superposition of  $\phi_1$ ,  $\phi_2$  and  $\phi_6$ , the corresponding Lagrange multipliers being  $\lambda_1 = 1$ ,  $\lambda_2 = \alpha = 0$ , and  $\lambda_6 = \beta = \dot{e}$ , respectively. All Lagrange multipliers for the second class constraints thus vanish. The Lagrange multipliers for the first class constraints can be arbitrary, they need not be fixed to  $\lambda_1 = 1$  and  $\lambda_6 = \dot{e}$ . In general, the total Hamiltonian is thus  $H_{\text{tot}} = \lambda_1 \phi_1 + \lambda_6 \phi_6$ , with arbitrary  $\lambda_1$ ,  $\lambda_6$ .

We are now interested in the behaviour of certain quantities on the constraint surface  $\Sigma$ . From the system of equations<sup>4</sup>  $\phi_1 = 0$ ,  $\phi_4 = 0$ ,  $\phi_5 = 0$ , valid on  $\Sigma$ , we can calculate  $p_\mu q^\mu/e$ ,  $mq^2/e^2$  and  $e^2 \pi^2/\mu$  in terms of  $p^2 \equiv p^\mu p_\mu = M^2$ , where  $M^2$  is a constant of motion.

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<sup>4</sup>Here  $\phi_1 = 0$  does not mean that  $\phi_1$  is strongly zero, it is zero on the constraint surface only (see footnote 3).

We obtain:

$$\frac{p_\mu q^\mu}{e} = \frac{M^2}{m}, \quad (81)$$

$$\frac{mq^2}{e^2} = \frac{m}{2} \left( \frac{3M^2}{m^2} - 1 \right), \quad (82)$$

$$\frac{e^2 \pi^2}{\mu} = m \left( \frac{M^2}{m^2} - 1 \right). \quad (83)$$

Because  $q^\mu = \dot{x}^\mu$  is a time-like vector, we have

$$0 \leq \frac{q^2}{e^2} \leq 1. \quad (84)$$

On the other hand,  $\pi_\mu$  is proportional to the acceleration, therefore we have

$$\pi^2 \equiv \pi^\mu \pi_\mu \leq 0. \quad (85)$$

The condition that (84) and (85) are simultaneously satisfied is

$$\frac{1}{3} \leq \frac{M^2}{m^2} < 1. \quad (86)$$

The Poisson brackets of the *first class constraints*  $\phi_1$  and  $\phi_6$  with all the constraints vanish. The Poisson brackets between the remaining constraints,  $\phi_{\bar{\alpha}}$ ,  $\bar{\alpha} = 2, 3, 4, 5$ , do not all vanish, therefore these are *second class constraints*. If we calculate the matrix  $C_{\bar{\alpha}\bar{\beta}} = \{\phi_{\bar{\alpha}}, \phi_{\bar{\beta}}\}$ , we obtain on the constraint surface that

$$C_{\bar{\alpha}\bar{\beta}} = \begin{pmatrix} 0 & 0 & \frac{3M^2}{m} - 2m & M^2 \\ 0 & 0 & M^2 & M^2 \\ -\left(\frac{3M^2}{m} - 2m\right) & -M^2 & 0 & 0 \\ -M^2 & -M^2 & 0 & 0 \end{pmatrix} \quad (87)$$

Its determinat is  $\det C_{\bar{\alpha}\bar{\beta}} = 4M^4(m^2 - M^2)^2$ . The reciprocal matrix is

$$C^{\bar{\alpha}\bar{\beta}} = \frac{1}{2(m^2 - M^2)} \begin{pmatrix} 0 & 0 & m & -1 \\ 0 & 0 & -1 & -\frac{2m^2 - 3M^2}{mM^2} \\ -m & 1 & 0 & 0 \\ 1 & \frac{2m^2 - 3M^2}{mM^2} & 0 & 0 \end{pmatrix} \quad (88)$$

Because of the presence of the second class constraints it is convenient to introduce the *Dirac brackets*, which are the projections of the Poisson brackets onto the constraint surface:

$$\{F, G\}_D = \{F, G\} - \{F, \phi_{\bar{\alpha}}\} C^{\bar{\alpha}\bar{\beta}} \{\phi_{\bar{\beta}}, G\}, \quad (89)$$

where  $F, G$  are phase space functions.

In particular, we have<sup>5</sup>

$$\{q^\mu, \phi_{\bar{\alpha}}\} = \left( q^\mu, ep^\mu, \frac{e^2\pi^\mu}{\mu}, 0 \right), \quad (90)$$

$$\{\pi^\mu, \phi_{\bar{\alpha}}\} = \left( -\pi^\mu, 0, -\frac{p^\mu}{e} + \frac{2mq^\mu}{e^2}, \frac{mp^\mu}{e} \right), \quad (91)$$

from which we can calculate the following Dirac brackets:

$$\{q^\mu, q^\nu\}_D = \frac{e^2m}{2\mu(m^2 - M^2)} \left( S^{\mu\nu} + \frac{e}{m}\pi^{[\mu}p^{\nu]} \right), \quad (92)$$

$$\{\pi^\mu, \pi^\nu\}_D = \frac{m^2}{(m^2 - M^2)e^2} \left( S^{\mu\nu} + \frac{e}{m}\pi^{[\mu}p^{\nu]} \right), \quad (93)$$

$$\{q^\mu, \pi^\nu\}_D = \eta^{\mu\nu} + \frac{1}{2(m^2 - M^2)} \left[ \frac{e^2m}{\mu}\pi^\mu\pi^\nu + 2 \left( 2 - \frac{m^2}{M^2} \right) p^\mu p^\nu + \frac{2m^2}{e^2}q^\mu q^\nu - \frac{2m}{e}(q^\mu p^\nu + q^\nu p^\mu) \right] \quad (94)$$

Here  $S^{\mu\nu} = q^\mu\pi^\nu - q^\nu\pi^\mu$  is *the spin tensor*. It is the orbital momentum in the  $q^\mu$ -space. Together with the orbital momentum in the  $x^\mu$ -space it forms *the total angular momentum*  $J^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu}$ , which is a conserved quantity for our action associated with the Lagrangian (56) [26, 13, 9].

The extra term  $(e/m)\pi^{[\mu}p^{\nu]}$  modifies the spin tensor  $S^{\mu\nu}$  into

$$\tilde{S}^{\mu\nu} = V^\mu\pi^\nu - V^\nu\pi^\mu, \quad (95)$$

where

$$V^\mu = q^\mu - \frac{ep^\mu}{m}, \quad (96)$$

so that we have

$$\{q^\mu, q^\nu\}_D = \{V^\mu, V^\nu\}_D = \frac{e^2m}{2\mu(m^2 - M^2)} \tilde{S}^{\mu\nu} \quad (97)$$

$$\{\pi^\mu, \pi^\nu\}_D = \frac{m^2}{e^2(m^2 - M^2)} \tilde{S}^{\mu\nu} \quad (98)$$

The spin tensor so modified is in fact the spin tensor subjected to the translation in the  $q^\mu$ -space, according to Eq. (96). In other words, it is the  $q^\mu$ -space orbital momentum translated by the vectors  $(e/m)p^\mu$ . From the equations of motion it is straightforward to derive that  $\tilde{S}^{\mu\nu}$  is a constant of motion,  $d\tilde{S}^{\mu\nu}/d\tau = 0$ , whereas  $dS^{\mu\nu}/d\tau = p^\mu q^\nu - p^\nu q^\mu$ . The Pauli-Lubanski pseudovector is the same for  $S^{\mu\nu}$  and  $\tilde{S}^{\mu\nu}$ :

$$S_\mu = \frac{1}{2M}\epsilon_{\mu\nu\rho\sigma}J^{\nu\rho}p^\sigma = \frac{1}{2M}\epsilon_{\mu\nu\rho\sigma}S^{\nu\rho}p^\sigma = \frac{1}{2M}\epsilon_{\mu\nu\rho\sigma}\tilde{S}^{\nu\rho}p^\sigma, \quad (99)$$

and it is a constant of motion.

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<sup>5</sup> By this notation we mean that  $\{q^\mu, \phi_2\} = q^\mu$ ,  $\{q^\mu, \phi_3\} = ep^\mu$ , etc..

By using (96) we will now simplify Eq. (94) as well. Since

$$V^\mu V^\nu = \left( q^\mu - \frac{ep^\mu}{m} \right) \left( q^\nu - \frac{ep^\nu}{m} \right) = q^\mu q^\nu - \frac{e}{m} (q^\mu p^\nu + q^\nu p^\mu) + \frac{e^2}{m^2} p^\mu p^\nu \quad (100)$$

and

$$2 \left( 2 - \frac{m^2}{M^2} \right) = 2 \left( 1 - \frac{m^2}{M^2} \right) + 2, \quad (101)$$

we obtain

$$\{q^\mu, \pi^\nu\}_D = \{V^\mu, \pi^\nu\}_D = \eta^{\mu\nu} - \frac{p^\mu p^\nu}{M^2} + \frac{1}{2(m^2 - M^2)} \left[ \frac{e^2 m^2}{\mu} \pi^\mu \pi^\nu + \frac{2m^2}{e^2} V^\mu V^\nu \right]. \quad (102)$$

Let us also calculate the following Dirac bracket:

$$\{q^\mu, S^{\mu\nu}\}_D = \{q^\mu, S^{\mu\nu}\} + \{q^\rho, \phi_{\bar{\alpha}}\} C^{\bar{\alpha}\bar{\beta}} \{S^{\mu\nu}, \phi_{\bar{\beta}}\}. \quad (103)$$

Using the relations,

$$\{S^{\mu\nu}, \phi_{\bar{\alpha}}\} = \left( 0, ep^{[\mu} \pi^{\nu]}, \frac{1}{e} p^{[\mu} q^{\nu]}, -\frac{m}{e} p^{[\mu} q^{\nu]} \right), \quad (104)$$

we obtain

$$\{q^\rho, S^{\mu\nu}\}_D = q^\mu \eta^{\nu\rho} - q^\nu \eta^{\mu\rho} + \frac{1}{2(m^2 - M^2)} \left[ \frac{e^3}{\mu} \pi^\rho p^{[\mu} \pi^{\nu]} + \frac{2(m^2 - 2M^2)}{M^2} p^\rho p^{[\mu} q^{\nu]} + \frac{2m}{e} q^\rho p^{[\mu} p^{\nu]} \right], \quad (105)$$

which is a rather complicated expression. Let us now calculate

$$\{V^\rho, \tilde{S}^{\mu\nu}\}_D = \{q^\rho, S^{\mu\nu}\}_D + \{q^\rho, \frac{e}{m} \pi^{[\mu} p^{\nu]}\}_D \quad (106)$$

Using Eqs. (88),(90) and

$$\{\pi^{[\mu} p^{\nu]}, \phi_{\bar{\alpha}}\} = \left( -\pi^{[\mu} p^{\nu]}, 0, \frac{2m}{e^2} q^{[\mu} p^{\nu]}, 0 \right), \quad (107)$$

we obtain

$$\begin{aligned} \{V^\rho, \tilde{S}^{\mu\nu}\}_D &= V^\mu \eta^{\nu\rho} - V^\nu \eta^{\mu\rho} + \frac{1}{2(m^2 - M^2)} \left( 1 + \frac{2m^2 - 3M^2}{M^2} \right) p^\rho p^{[\mu} q^{\nu]} \\ &= V^\mu \eta^{\nu\rho} - V^\nu \eta^{\mu\rho} + \frac{1}{M^2} p^\rho p^{[\mu} q^{\nu]} \\ &= V^\mu \left( \eta^{\nu\rho} - \frac{1}{M^2} p^\nu p^\rho \right) - V^\nu \left( \eta^{\mu\rho} - \frac{1}{M^2} p^\mu p^\rho \right), \end{aligned} \quad (108)$$

where we have put  $p^{[\mu} q^{\nu]} = p^{[\mu} (q^{\nu]} - (e/m)p^{\nu]}) = p^{[\mu} V^{\nu]}$ .

Similarly, we obtain

$$\{\pi^\rho, \tilde{S}^{\mu\nu}\}_D = \pi^\mu \left( \eta^{\nu\rho} - \frac{1}{M^2} p^\nu p^\rho \right) - \pi^\nu \left( \eta^{\mu\rho} - \frac{1}{M^2} p^\mu p^\rho \right) \quad (109)$$

Using (96), the constraint (71) can be written in the form

$$V^\mu p_\mu = 0. \quad (110)$$

The algebra of the Dirac brackets (97),(98), (108) and (109) resembles the algebra of the quantum commutators that are satisfied by the Dirac matrices. A difference is in Eqs.(108),(109), where the Minkowski metric  $\eta^{\nu\rho}$  is changed into the modified metric  $\eta^{\mu\rho} - p^\mu p^\rho / M^2$ .

*Equations of motion in terms of the Dirac brackets*

If in eqs. (58)–(63) we replace the Poisson brackets with the Dirac brackets, we obtain the same equations of motion. So we have

$$\dot{q}^\rho = \{q^\rho, H\}_D = \{q^\rho, H\} + \{q^\rho, \Phi_{\bar{\alpha}}\} C^{\bar{\alpha}\bar{\beta}} \{H, \Phi_{\bar{\beta}}\} = \{q^\rho, H\}, \quad (111)$$

where we have taken into account that  $\{H, \Phi_{\bar{\beta}}\} = 0$ .

To check that we have correctly computed the Dirac brackets (92)–(110), it is instructive to derive

$$\begin{aligned} \dot{q}^\rho = \{q^\rho, H\}_D &= p_\mu \{q^\rho, q^\mu\}_D - \frac{e^3}{4\mu} \{q^\rho, \pi^2\}_D - \frac{m}{2e} \{q^\rho, q^2\}_D \\ &+ \frac{\beta}{e} \{q^\rho, \pi^\mu\}_D q_\mu + \frac{\beta}{e} \{q^\rho, q^\mu\}_D \pi_\mu + \beta \{q^\rho, p_e\}_D, \end{aligned} \quad (112)$$

and the analogous equations for the other dynamical variables. Besides Eqs. (92)–(98), we also need

$$\{q^\rho, p_e\}_D = \{q^\rho, \Phi_{\bar{\alpha}}\} C^{\bar{\alpha}\bar{\beta}} \{p_e, \Phi_{\bar{\beta}}\} \quad (113)$$

Using (90) and

$$\{p_e, \phi_{\bar{\alpha}}\} = (0, 0, -\frac{3M^2}{em} + \frac{2m}{e}, -\frac{M^2}{e}), \quad (114)$$

we have

$$\{q^\rho, p_e\}_D = \frac{q^\rho}{e}. \quad (115)$$

If we explicitly calculate the terms in Eq. (112) by using (92)–(98) and (115), we obtain

$$\dot{q}^\rho = -\frac{e^3}{2\mu} \pi^\rho + \frac{\beta}{e} q^\rho, \quad (116)$$

which is in agreement with Eq. (60). In deriving the latter equation we have taken into account that for the metric  $g^{\rho\mu} = \eta^{\rho\mu} - p^\rho p^\mu / M^2$  we have

$$g^{\rho\mu} \pi_\mu = \eta^{\rho\mu} \pi_\mu - p^\rho p^\mu \pi_\mu / M^2 = \eta^{\rho\mu} \pi_\mu = \pi^\rho, \quad (117)$$

$$g^{\rho\mu} q_\mu = q^\rho - p^\rho \frac{p^\mu q_\mu}{M^2} = q^\rho - p^\rho \frac{e}{m} = V^\rho, \quad (118)$$

where we have used the constraints  $\Phi_3$  and  $\Phi_5$ , which give  $\pi^\mu p_\mu = 0$  and  $mp^\mu q_\mu / e = M^2$ .



We will now introduce the new dynamical variables,

$$\Gamma^\mu = \frac{q^\mu}{e} - \frac{p^\mu}{m} = \frac{V^\mu}{e}, \quad \Pi^\mu = e\pi^\mu, \quad (119)$$

which are invariant under reparametrizations  $\tau \rightarrow \tau' = h(\tau)$ . They satisfy the following relations:

$$\tilde{S}^{\mu\nu} = \frac{2\mu(m^2 - M^2)}{m} \{\Gamma^\mu, \Gamma^\nu\}_D \quad (120)$$

$$\{\Gamma^\rho, \tilde{S}^{\mu\nu}\}_D = \Gamma^\mu g^{\nu\rho} - \Gamma^\nu g^{\mu\rho} \quad (121)$$

$$\tilde{S}^{\mu\nu} = \frac{(m^2 - M^2)}{m^2} \{\Pi^\mu, \Pi^\nu\}_D \quad (122)$$

$$\{\Pi^\rho, \tilde{S}^{\mu\nu}\}_D = \Pi^\mu g^{\nu\rho} - \Pi^\nu g^{\mu\rho} \quad (123)$$

$$\{\Gamma^\mu, \Pi^\nu\} = g^{\mu\nu} + \frac{m}{m^2 - M^2} \left( \frac{\Pi^\mu \Pi^\nu}{2\mu} + m\Gamma^\mu \Gamma^\nu \right). \quad (124)$$

The second class constraints now read as

$$\phi_2 = \Pi^\mu \left( \Gamma^\mu + \frac{p^\mu}{m} \right) \quad (125)$$

$$\phi_3 = \Pi^\mu p_\mu \quad (126)$$

$$\phi_4 = \frac{\Pi^\mu \Pi_\mu}{2\mu} - m\Gamma^\mu \Gamma_\mu - p_\mu \Gamma^\mu \quad (127)$$

$$\phi_5 = -p_\mu \Gamma^\mu, \quad (128)$$

which can be transformed into the following set of constraints:

$$\psi_2 = \phi_2 - \frac{\phi_3}{m} = \Pi^\mu \Gamma_\mu \quad (129)$$

$$\psi_3 = \phi_3 = \Pi^\mu p_\mu \quad (130)$$

$$\psi_4 = \phi_4 + \phi_5 = \frac{\Pi^\mu \Pi_\mu}{2\mu} - m\Gamma^\mu \Gamma_\mu \quad (131)$$

$$\psi_5 = -\phi_5 = p_\mu \Gamma^\mu. \quad (132)$$

In terms of the new variables, the Hamiltonian (57) (for  $\alpha = 0$ ) reads

$$H = e \left[ -\frac{m}{2} \Gamma^\mu \Gamma_\mu - \frac{1}{4\mu} \Pi^\mu \Pi_\mu + \frac{1}{2m} (p^2 - m^2) \right] + \frac{\beta}{e} (ep_e + \Pi_\mu \Gamma^\mu) \quad (133)$$

It is a superposition of the first class constraints

$$\phi_1 = -\frac{m}{2} \Gamma^\mu \Gamma_\mu - \frac{1}{4\mu} \Pi^\mu \Pi_\mu + \frac{1}{2m} (p^2 - m^2) \quad (134)$$

$$\phi_6 = ep_e + \Pi_\mu \Gamma^\mu \quad (135)$$

As an example, let us compute the following quantity, which is invariant under reparametrizations of  $\tau$ :

$$\frac{\dot{\Gamma}^\rho}{e} = \frac{1}{e} \frac{d}{d\tau} \left( \frac{q^\rho}{e} \right) = \frac{1}{e} \{\Gamma^\rho, H\}_D \quad (136)$$

Because  $\{\Gamma^\rho, p^2\}_D = 0$  and  $\{\Gamma^\rho, \frac{\beta}{e}(ep_e + \Pi_\mu \Gamma^\mu)\}_D = 0$ , we have

$$\begin{aligned}
\frac{\dot{\Gamma}^\rho}{e} &= \{\Gamma^\rho, -\frac{m}{2}\Gamma^\mu \Gamma_\mu - \frac{1}{4\mu}\Pi^\mu \Pi_\mu\}_D = -\frac{m}{2}\{\Gamma^\rho, \Gamma^\mu \Gamma_\mu\}_D - \frac{1}{4\mu}\{\Pi^\mu \Pi_\mu\}_D \\
&= -\frac{m}{2}[\Gamma_\mu\{\Gamma^\rho, \Gamma^\mu\}_D + \{\Gamma^\rho, \Gamma^\mu\}_D \Gamma_\mu] - \frac{1}{4\mu}[\Pi_\mu\{\Gamma^\rho, \Pi^\mu\}_D + \{\Gamma^\rho, \Pi^\mu\}_D \Pi_\mu] \\
&= -\frac{m}{2} \frac{m}{2\mu(m^2 - M^2)}(\Gamma_\mu \tilde{S}^{\rho\mu} + \tilde{S}^{\rho\mu} \Gamma_\mu) \\
&\quad - \frac{2}{4\mu} \Pi_\mu \left[ g^{\rho\mu} + \frac{1}{2(m^2 - M^2)} \left( \frac{m}{\mu} \Pi^\rho \Pi^\mu + 2m^2 \Gamma^\rho \Gamma^\mu \right) \right] \\
&= -\frac{m}{2} \frac{2m}{2\mu(m^2 - M^2)} \Gamma_\mu (\Gamma^\rho \Pi^\mu - \Gamma^\mu \Pi^\rho) - \frac{1}{2\mu} \left[ \Pi_\mu g^{\rho\mu} + \frac{1}{2(m^2 - M^2)} \frac{m}{\mu} \Pi^\rho \Pi_\mu \Pi^\mu \right] \\
&= -\frac{1}{2\mu} \Pi^\rho + \frac{m \Pi^\rho}{2\mu(m^2 - M^2)} + \left( m \Gamma^\mu \Gamma_\mu - \frac{\Pi^\mu \Pi_\mu}{2\mu} \right). \tag{137}
\end{aligned}$$

In the last term of the latter equation we have the expression

$$m \Gamma^\mu \Gamma_\mu - \frac{\Pi^\mu \Pi_\mu}{2\mu} = m \left( \frac{q^\mu}{e} - \frac{p^\mu}{m} \right) \left( \frac{q_\mu}{e} - \frac{p_\mu}{m} \right) - \frac{e^2 \pi^2}{2\mu} = 0, \tag{138}$$

which vanishes because of the constraints  $\Phi_4 = 0$  and  $\Phi_5 = 0$ . Therefore, Eq. (137) gives

$$\frac{\dot{\Gamma}^\rho}{e} = \{\Gamma^\rho, -\frac{m}{2}\Gamma^\mu \Gamma_\mu - \frac{1}{4\mu}\Pi^\mu \Pi_\mu\}_D = -\frac{1}{2\mu} \Pi^\rho. \tag{139}$$

The latter result can be much quicker obtained if instead of the Dirac brackets we use the Poisson brackets, which satisfy

$$\{\Gamma^\rho, \Gamma_\mu\} = 0, \quad \{\Gamma^\rho, \Pi^\mu\} = \eta^{\rho\mu}. \tag{140}$$

Then it is straightforward to verify that

$$\frac{\dot{\Gamma}^\rho}{e} = \{\Gamma^\rho, -\frac{m}{2}\Gamma^\mu \Gamma_\mu - \frac{1}{4\mu}\Pi^\mu \Pi_\mu\} = -\frac{1}{2\mu} \Pi^\rho. \tag{141}$$

Similarly, we obtain

$$\frac{\dot{\Pi}^\rho}{e} = m \Gamma^\rho. \tag{142}$$

Together, Eqs. (141), (142) give

$$\frac{d^2 \Gamma^\rho}{ds^2} + \omega^2 \Gamma^\rho = 0, \tag{143}$$

where  $\omega^2 = m^2/(2\mu)$ , and  $ds = ed\tau$ .

However, the longer procedure with the Dirac brackets provides a test for the correctness of the computed Dirac brackets. It will also guide us in the quantized theory in which the Dirac brackets will be replaced by commutators.

## 4 Quantization

We will now consider the quantization based on the Dirac brackets. According to this procedure, the classical quantities become the operators that satisfy the commutation relations corresponding to the Dirac bracket relations [28]. In the following, we will consider the quantum versions of the Dirac bracket relations of the previous section.

The first class constraints  $\phi_1$  and  $\phi_6$  act as restrictions on the Hilbert space of states according to

$$\phi_1|\psi\rangle = 0, \quad \phi_6|\psi\rangle = 0, \quad (144)$$

where the states satisfying the above equations are *physical states*.

The second class constraints commute with all phase space operators and any function of them,

$$[\phi_{\bar{\alpha}}, f(x^\mu, p_\mu, q^\mu, \pi_\mu, e, p_e)] = 0. \quad (145)$$

Therefore, they are *c*-numbers that satisfy

$$\phi_{\bar{\alpha}} = 0, \quad \bar{\alpha} = 2, 3, 4, 5. \quad (146)$$

Let us introduce the operators

$$\Gamma^\mu = \frac{V^\mu}{e}, \quad \Pi^\mu = e\pi^\mu, \quad (147)$$

whose classical analogs, satisfying the constraints (129)–(132), were introduced in Eq. (119). The corresponding quantum constraints have the same form, except that  $\psi_2$  is now symmetrized according to

$$\psi_2 = \frac{1}{2}(\Pi^\mu \Gamma_\mu + \Gamma_\mu \Pi^\mu) \quad (148)$$

We will now consider the quantum version of the Dirac bracket relations (120)–(124), satisfied by  $\Gamma^\mu$  and  $\Pi^\mu$ . Writing

$$\frac{2\mu(m^2 - p^2)}{m} \equiv \frac{\rho^2}{4}, \quad \frac{(m^2 - p^2)}{m^2} \equiv \frac{\tilde{\rho}^2}{4} \quad (149)$$

$$g^{\mu\nu} = \eta^{\mu\nu} - \frac{1}{p^2} p^\mu p^\nu \quad (150)$$

we have

$$\tilde{S}^{\mu\nu} = j_1 \frac{\rho^2}{4} [\Gamma^\mu, \Gamma^\nu], \quad (151)$$

$$[\Gamma^\rho, \tilde{S}^{\mu\nu}] = j_2 (\Gamma^\mu g^{\nu\rho} - \Gamma^\nu g^{\mu\rho}). \quad (152)$$

$$\tilde{S}^{\mu\nu} = j_3 \frac{\tilde{\rho}^2}{4} [\Pi^\mu, \Pi^\nu], \quad (153)$$

$$[\Pi^\rho, \tilde{S}^{\mu\nu}] = j_4 (\Pi^\mu g^{\nu\rho} - \Pi^\nu g^{\mu\rho}). \quad (154)$$

The quantum version of the Dirac bracket (102) is

$$[\Gamma^\mu, \Pi^\nu] = j \left( g^{\mu\nu} + \frac{4}{\rho^2} \Pi^\mu \cdot \Pi^\nu + \frac{4}{\tilde{\rho}^2} \Gamma^\mu \cdot \Gamma^\nu \right), \quad (155)$$

where the dot denotes the symmetrized product, e.g.,  $\Gamma^\mu \cdot \Gamma^\nu = \frac{1}{2}(\Gamma^\mu \Gamma^\nu + \Gamma^\nu \Gamma^\mu)$ .

In addition we also have the relation

$$\tilde{S}^{\mu\nu} = \Gamma^\mu \Pi^\nu - \Gamma^\nu \Pi^\mu \quad (156)$$

which corresponds to the definition  $\tilde{S}^{\mu\nu} = V^\mu \pi^\nu - V^\nu \pi^\mu$  of the modified spin tensor  $\tilde{S}^{\mu\nu}$ .

The relations (151)–(156) are the quantum counterpart of the classical equations (120)–(124),(95), whose structure is more complicated than that of the usual relations, such as  $\{x^\mu, p_\nu\} = \delta^\mu_\nu$ ,  $\{x^\mu, x^\nu\} = 0$ ,  $\{p_\mu, p_\nu\} = 0$ . The latter Poisson bracket relations can be replaced by the operator commutation relations  $[\hat{x}^\mu, \hat{p}_\nu] = \delta^\mu_\nu$ ,  $[\hat{x}^\mu, \hat{x}^\nu] = 0$ ,  $[\hat{p}_\mu, \hat{p}_\nu] = 0$ , that are satisfied by the operators represented as  $\hat{x}^\mu = x^\mu$ ,  $\hat{p}_\mu = -\partial_\mu$ . However, the operator  $\hat{p}_\mu$  so defined, is not Hermitian, therefore we make the replacement  $\hat{p}_\mu = -\partial_\mu \rightarrow \hat{p}_\mu = -i\partial_\mu$ , so that the quantum commutator becomes  $[\hat{x}^\mu, \hat{p}_\nu] = i\delta^\mu_\nu$ . In the case of the rather complicated Poisson bracket system (97),(98),(108),(110),(105), or equivalently, (120)–(124), we cannot a priori expect that the replacement  $\{, \}_D \rightarrow \frac{1}{i}[\, , \,]$  will work. Therefore, in the commutation relations (151)–(155) we have introduced the quantities  $j_1, j_2, j_3, j_4$  and  $j$ , which will be determined in the process of finding a consistent representation for  $\Gamma^\mu$ ,  $\Pi^\mu$ , satisfying those commutation relations, as well as the definition (156) of  $\tilde{S}^{\mu\nu}$ .

The solution to the system (151)–(154) are the operators  $\Gamma^\mu$ ,  $\Pi^\mu$  that satisfy the Clifford algebra relations with the modified metric:

$$\frac{j_1}{j_2} \frac{1}{2} (\Gamma^\mu \Gamma^\nu + \Gamma^\nu \Gamma^\mu) = -\rho^{-2} g^{\mu\nu}, \quad (157)$$

$$\frac{j_3}{j_4} \frac{1}{2} (\Pi^\mu \Pi^\nu + \Pi^\nu \Pi^\mu) = -\tilde{\rho}^{-2} g^{\mu\nu}. \quad (158)$$

The metric  $g^{\mu\nu} = \eta^{\mu\nu} - p^\mu p^\nu / p^2$  can be transformed from the Minkowski metric  $\eta^{\mu\nu} = \text{diag}(1, -1, -1, -1)$  according to

$$g^{\mu\nu} = \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x'^\nu}{\partial x^\beta} \eta^{\alpha\beta}, \quad (159)$$

where

$$x' = \left( x^\mu - \frac{p^\mu x^\rho p_\rho}{p^2} \right), \quad \frac{\partial x'^\mu}{\partial x^\alpha} = \left( \delta^\mu_\alpha - \frac{p^\mu p_\alpha}{p^2} \right). \quad (160)$$

We see that because of the factors  $\rho^2$  and  $\tilde{\rho}^2$  in Eqs. (157) and (158), the above coordinate transformation is accompanied by the corresponding dilatation.

Eqs. (151),(152) (and (153),(154)) are similar to the equations that come from the relativistic covariance of the Dirac equation, which gives

$$[\gamma^\rho, \sigma^{\mu\nu}] = 2i(\eta^{\rho\mu} \gamma^\nu - \eta^{\rho\nu} \gamma^\mu), \quad (161)$$

where  $\sigma^{\mu\nu}$  are generators of Lorentz transformations. The latter equation is satisfied by

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]. \quad (162)$$

In the theory of the Dirac equation, the Clifford algebra relations for the objects  $\gamma^\mu$ , namely,

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu}, \quad (163)$$

are already given, whereas the relation (162) for the generators  $\sigma^{\mu\nu}$  is computed from (161) and (163). In our procedure, on the contrary, we arrived at the commutation relations (151), (152), which are very similar to Eqs. (162), (161), respectively, and a question was, how to represent the operators  $\Gamma^\mu$ . We have found by direct calculation that  $\Gamma^\mu$  satisfy the Clifford algebra relations (157). Analogous hold for the operators  $\Pi^\mu$ , satisfying Eqs. (153), (154), (158).

#### 4.1 Schrödinger picture

In the Schrödinger picture, the operators do not evolve in time.

If  $j_1 = j_2$ , then Eq. (157) is satisfied by

$$\Gamma^\mu = \frac{e_q}{\rho} \frac{\partial x'^\mu}{\partial x^\alpha} \gamma^\alpha = \frac{e_q}{\rho} \left( \gamma^\mu - \frac{p^\mu p_\alpha \gamma^\alpha}{p^2} \right), \quad (164)$$

where  $e_q^2 = -1$ , whereas  $\gamma^\mu$  satisfy the Clifford algebra relations (163).

Similarly, if  $j_3 = j_4$ , then Eq. (158) is satisfied by<sup>6</sup>

$$\Pi^\mu = -\frac{e_\pi}{\tilde{\rho}} \frac{\partial x'^\mu}{\partial x^\alpha} \gamma^\alpha = -\frac{e_\pi}{\tilde{\rho}} \left( \gamma^\mu - \frac{p^\mu p_\alpha \gamma^\alpha}{p^2} \right), \quad (165)$$

where  $e_\pi^2 = -1$ . We assume that  $e_q$  and  $e_\pi$  commute with  $\gamma^\mu$ ,

The quantities

$$\alpha^\mu = \gamma^\mu - \frac{p^\mu p_\alpha \gamma^\alpha}{p^2} \quad (166)$$

are projections of  $\gamma^\mu$  onto the 3-dimensional hypersurface that is orthogonal to the direction of the 4-momentum  $p^\mu$ . Thus, vectors  $\alpha^\mu$  are “spatial” parts of vectors  $\gamma^\mu$ . They satisfy the Clifford algebra relations

$$\alpha^\mu \cdot \alpha^\nu \equiv \frac{1}{2}(\alpha^\mu \alpha^\nu + \alpha^\nu \alpha^\mu) = g^{\mu\nu} = \eta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \quad (167)$$

In view of the above relations (164)–(166), the spin operator determined by Eq. (151) and (153), becomes

$$\tilde{S}^{\mu\nu} = -\frac{j_1}{4}[\alpha^\mu, \alpha^\nu] = -\frac{j_3}{4}[\alpha^\mu, \alpha^\nu], \quad (168)$$

from which it follows that  $j_1 = j_3$ .

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<sup>6</sup> We take the minus sign in front of  $e_\pi$  for later convenience..

On the other hand, according to Eq. (156), the spin tensor is

$$\tilde{S}^{\mu\nu} = \Gamma^\mu \Pi^\nu - \Gamma^\nu \Pi^\mu = -\frac{e_q e_\pi}{\rho \tilde{\rho}} [\alpha^\mu, \alpha^\nu]. \quad (169)$$

By comparing Eqs. (168) and (169) we obtain the relations

$$j_1 = e_q e_\pi \quad (170)$$

$$\rho \tilde{\rho} = 4, \quad \text{i.e.,} \quad m^2 - p^2 - m \sqrt{\frac{m}{2\mu}} = 0, \quad (171)$$

where  $p^2 \equiv p^\mu p_\mu$  is the squared momentum operator.

Using (164),(165), we have

$$[\Gamma^\mu, \Pi^\nu] = -\frac{1}{\rho \tilde{\rho}} [e_q \alpha^\mu, e_\pi \alpha^\nu] = -\frac{1}{\rho \tilde{\rho}} (e_q e_\pi \alpha^\mu \alpha^\nu - e_\pi e_q \alpha^\nu \alpha^\mu). \quad (172)$$

On the other hand, according to (155), the commutator is

$$[\Gamma^\mu, \Pi^\nu] = j \left( g^{\mu\nu} + \frac{4}{\rho^2} \Pi^\mu \cdot \Pi^\nu + \frac{4}{\tilde{\rho}^2} \Gamma^\mu \cdot \Gamma^\nu \right) = j g^{\mu\nu} \left( 1 - \frac{8}{\rho^2 \tilde{\rho}^2} \right) = j \frac{g^{\mu\nu}}{2}, \quad (173)$$

where in the last step we used Eq. (171).

The commutators in Eqs. (172) and (173) must be the same. This is possible, if  $e_q$  and  $e_\pi$  are the numbers satisfying, besides

$$e_q^2 = -1, \quad e_\pi^2 = -1, \quad (174)$$

also the relation

$$e_q e_\pi + e_\pi e_q = 0. \quad (175)$$

This means that  $e_q, e_\pi$  are elements of the Clifford algebra  $Cl(0, 2)$ . The latter notation means that the signature of the vector space spanned by the basis vectors  $e_q$  and  $e_\pi$  is  $(--)$  or  $(0, 2)$ .<sup>7</sup>

Taking into account Eq. (175) in Eq. (172), we obtain

$$[\Gamma^\mu, \Pi^\nu] = -\frac{1}{\rho \tilde{\rho}} e_q e_\pi (\alpha^\mu \alpha^\nu + \alpha^\nu \alpha^\mu) = -e_q e_\pi \frac{2g^{\mu\nu}}{\rho \tilde{\rho}}. \quad (176)$$

By using in the latter equation the relation (171), we arrive at the same result as in Eq. (173), if we put

$$j = -e_q e_\pi. \quad (177)$$

Altogether, we have

$$j_1 = j_2 = j_3 = j_4 = -j = e_q e_\pi. \quad (178)$$

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<sup>7</sup>In general,  $Cl(p, q)$  is the Clifford algebra of a vector space with signature  $(p, q)$ . In particular, the Clifford algebra of spacetime is  $Cl(1, 3)$ .

The square is  $j^2 = (e_q e_\pi)^2 = e_q e_\pi e_q e_\pi = -e_q e_\pi^2 e_q = e_q^2 = -1$ . We see that  $j$ , which is a bivector of the Clifford algebra  $Cl(0, 2)$ , behaves as imaginary unit.

Thus we have found that the system of the operator equations (151)–(156) is solved by the operators  $\Gamma^\mu$ ,  $\Pi^\mu$ , defined according to Eqs. (164)–(165), provided that the quantities  $j_1, j_2, j_3, j_4$  and  $j$  satisfy (178).

From (168)–(166) we have the following expression for the spin operators

$$\tilde{S}^{\mu\nu} = -\frac{j_1}{4}[\gamma^\alpha, \gamma^\beta]g_\alpha^\mu g_\beta^\nu, \quad (179)$$

where  $g_\alpha^\mu = \delta_\alpha^\mu - \frac{p_\alpha p^\mu}{p^2}$ . The Pauli-Lubanski operator in 4-dimensions is

$$S_\mu = \frac{1}{2M}\epsilon_{\mu\nu\rho\sigma}\tilde{S}^{\nu\rho}p^\sigma = \frac{1}{2M}\epsilon_{\mu\nu\rho\sigma}S^{\nu\rho}p^\sigma, \quad (180)$$

where

$$S^{\nu\rho} = -\frac{j_1}{4}[\gamma^\nu, \gamma^\rho] \quad (181)$$

is the usual spin operator, given in terms of the Dirac gammas. The eigenvalues of its square,  $S^\mu S_\mu$ , are  $s(s+1) = 3/4$ , implying  $s = \frac{1}{2}$ .

## 4.2 Heisenberg picture

In the quantized theory the classical Hamiltonian is replaced by the corresponding operator. Instead of (136), we have the Heisenberg equations of motion

$$\frac{\dot{\Gamma}^\rho}{e} = -j\frac{1}{e}[\Gamma^\rho, H] = -j[\Gamma^\rho, -\frac{m}{2}\Gamma^\mu\Gamma_\mu - \frac{1}{4\mu}\Pi^\mu\Pi_\mu]. \quad (182)$$

In the following we will use the relations

$$[\Gamma^\rho, \Gamma^\mu\Gamma_\mu] = [\Gamma^\rho, \Gamma^\mu]\Gamma_\mu + \Gamma_\mu[\Gamma^\rho, \Gamma^\mu], \quad (183)$$

$$[\Gamma^\rho, \Pi^\mu\Pi_\mu] = [\Gamma^\rho, \Pi^\mu]\Pi_\mu + \Pi_\mu[\Gamma^\rho, \Pi^\mu]. \quad (184)$$

Inserting

$$[\Gamma^\rho, \Gamma^\mu] = -\frac{4}{j\rho^2}\tilde{S}^{\rho\mu}, \quad j_1 = -j = e_q e_\pi, \quad (185)$$

and

$$[\Gamma^\rho, \Pi^\mu] = j\left(g^{\rho\mu} + \frac{4}{\rho^2}\Pi^\rho \cdot \Pi^\mu + \frac{4}{\rho^2}\Gamma^\rho \cdot \Gamma^\mu\right) \equiv j\tilde{G}^{\rho\mu} \quad (186)$$

into (182), we obtain

$$\frac{\dot{\Gamma}^\rho}{e} = -\frac{m}{2}(\Gamma_\mu\tilde{S}^{\rho\mu} + \tilde{S}^{\rho\mu}\Gamma_\mu)\frac{4}{j\rho^2} + \frac{1}{4\mu}(\Pi_\mu\tilde{G}^{\rho\mu} + \tilde{G}^{\rho\mu}\Pi_\mu), \quad (187)$$

which, after using the definition (156) of  $\tilde{S}^{\rho\mu}$  and the constraint (148), becomes

$$\frac{\dot{\Gamma}^\rho}{e} = -\frac{1}{2\mu}\left(2\Pi_\mu g^{\rho\mu} + \frac{4}{\rho^2}\Pi_\mu\Pi^\rho\Pi^\mu\right) - \frac{2m}{\rho^2}(\Gamma_\mu\Gamma^\rho\Pi^\mu - \Gamma_\mu\Pi^\rho\Gamma^\mu + \Pi^\mu\Gamma^\rho\Gamma_\mu). \quad (188)$$

The latter equation, apart from the order of operators, matches the classical equation (137). By using (157), (158) and (186), we can reverse the order of operators in the products  $\Pi^\rho \Pi^\mu$ ,  $\Gamma^\rho \Pi^\mu$ ,  $\Pi^\rho \Gamma^\mu$  and  $\Gamma^\rho \Gamma_\mu$ , at the expense of acquiring certain extra terms. Using also the operator version of the constraint (131), we arrive at the equation

$$\frac{\dot{\Gamma}^\rho}{e} = -\frac{1}{2\mu}\Pi^\rho + \frac{4}{\mu\rho^2\tilde{\rho}^2}\Pi^\rho - j\frac{4m}{\rho^2}\tilde{G}^{\rho\mu}\Gamma_\mu \quad (189)$$

The latter equation, in comparison with the corresponding classical equation (139), has two extra terms.

Similarly, we obtain

$$\frac{\dot{\Pi}^\rho}{e} = m\Gamma^\rho - \frac{8m}{\rho^2\tilde{\rho}^2}\Gamma^\rho + j\frac{2}{\mu\tilde{\rho}^2}\tilde{G}^{\rho\mu}\Pi_\mu. \quad (190)$$

In the absence of the extra terms, Eqs., (189), (190) would give

$$\frac{d^2\Gamma^\rho}{ds^2} + \omega^2\Gamma^\rho = 0, \quad \Pi^\rho = -2\mu\frac{d\Gamma^\rho}{ds}, \quad ds = e d\tau, \quad (191)$$

with the solution

$$\Gamma^\rho = a^\rho \cos \omega s + b^\rho \sin \omega s, \quad (192)$$

$$\Pi^\rho = -2\mu\omega(-a^\rho \sin \omega s + b^\rho \cos \omega s), \quad (193)$$

where

$$\omega = \sqrt{\frac{m}{2\mu}}. \quad (194)$$

At  $s = 0$ , we have

$$\Gamma^\rho(0) = a^\rho, \quad \Pi^\rho(0) = -2\mu\omega b^\rho \quad (195)$$

Comparison with Eqs. (164), (165) gives

$$a^\rho = \frac{e_q \alpha^\rho}{\rho}, \quad b^\rho = \frac{e_\pi \alpha^\rho}{\rho}, \quad (196)$$

where  $\alpha^\rho$  is defined in Eq. (166).

If we insert the expressions (196) for  $a^\rho$  and  $b^\rho$  into Eqs. (192), (193), we obtain the following relations between  $\Pi^\rho$  and  $\Gamma^\rho$ :

$$\Pi^\rho = -2\mu\omega e_q e_\pi \Gamma^\rho. \quad (197)$$

We see that the bivector  $e_q e_\pi$  performs a  $\pi/2$  rotation in phase space, and thus, up to the factor  $2\mu\omega$ , exchanges  $\Pi^\rho$  and  $\Gamma^\rho$ .

In deriving Eq. (197), we assumed that the second and the third term in the equations of motion (189), (190) cancel out. According to Eq. (189) this is the case if

$$\Pi^\rho = j\mu m \tilde{\rho}^2 \Gamma_\mu \tilde{G}^{\rho\mu}, \quad (198)$$



and according to eq. (190) if

$$\Gamma^\rho = -\frac{j\rho^2}{4\mu m}\tilde{G}^{\rho\mu}\Pi_\mu. \quad (199)$$

Using Eqs. (186),(171),(157),(158), we obtain

$$\tilde{G}^{\rho\mu} = \frac{g^{\rho\mu}}{2}, \quad (200)$$

and

$$\Pi^\rho = j\sqrt{2m\mu}\Gamma^\rho, \quad (201)$$

which gives the relations (197), if we insert  $j = -e_q e_\pi$  and  $\omega = \sqrt{m/2\mu}$ . The cancelation of the second and the third term in Eqs. (189),(190) is thus consistent with the equations of motion (191) that give the relation (197).

The Heisenberg equations of motion (187) are thus consistent with the representation of operators (164),(165), evolving according to Eqs. (192),(193).

### 4.3 The physical states

The explicit form of Eq. (144) is the quantum analog of the constraints  $\phi_1, \phi_6$  given in Eqs. (69),(73), or equivalently, in Eqs. (134),(135). The physical states thus satisfy the equation

$$\left[ -\frac{m}{2}\Gamma^\mu\Gamma_\mu - \frac{1}{4\mu}\Pi^\mu\Pi_\mu + \frac{1}{2m}(p^2 - m^2) \right] |\Psi\rangle = 0, \quad (202)$$

$$(ep_e + \Pi_\mu\Gamma^\mu)|\Psi\rangle = 0. \quad (203)$$

Because of the second class constraint  $\psi_2 = \Pi_\mu\Gamma^\mu = 0$ , Eq. (203) gives  $p_e|\Psi\rangle = 0$ , which is in agreement with the classical equation  $p_e = 0$ .

If we use the Clifford algebra relations (157),(158) and Eqs. (149),(150), we obtain

$$m\Gamma^\mu\Gamma_\mu = -\frac{(D-1)m^2}{8\mu(m^2 - p^2)} = \frac{\Pi^\mu\Pi_\mu}{2\mu}, \quad (204)$$

where  $D$  is the dimension of spacetime. On the other hand, we also have the relation (171) which must now be written as a condition on physical states:

$$\left( m^2 - p^2 - m\sqrt{\frac{m}{2\mu}} \right) |\psi\rangle. \quad (205)$$

A physical state is a superposition of the basis states,  $|x, \alpha\rangle = |x\rangle|\alpha\rangle$ , which can be written as the product of the coordinate states  $|x\rangle$  and the spinor states  $|\alpha\rangle$ . With respect to the spin states  $|\alpha\rangle$ , the operators  $p^2$  is diagonal, which justifies usage of the operator equation (171). But now we also take into account the coordinate states  $|x\rangle$ , in which the operator  $p^2$  is not diagonal. Therefore, we have to use equation (205). Usage of Eq. (171) was just a short cut, valid for the matrix elements of  $p^2$  between the spinor state  $|\alpha\rangle$ , and so we represented  $p^2 \rightarrow \langle\alpha|p^2|\alpha'\rangle = p^2\delta_{\alpha\alpha'}$ .

If we plug Eqs. (204) into the equation of state (202), and take into account (205), we find that the terms do not cancel out, but that there remain the residual terms, giving

$$\frac{1}{4}(D-3)\omega|\Psi\rangle = 0, \quad (206)$$

where

$$\omega = \sqrt{\frac{m}{2\mu}}. \quad (207)$$

The above equation is satisfied if the dimension of spacetime is  $D = 3$ . Such a restriction on dimensionality of spacetime is very undesirable, and ruins our construction in which we assumed at least 4-dimensional spacetime. Alternatively, Eq. (206) is satisfied if  $\omega$ , which corresponds to the classical frequency of circular motion, vanishes. Such a restriction, of course, also invalidates our model.

A possible solution to such an anomaly in the quantized theory is in introducing yet another time-like dimension into our model. Let us generalize the string action to include a  $(D+2)^{\text{th}}$  dimension which, as the  $(D+1)^{\text{th}}$  one, is time-like:

$$I = \frac{T}{2} \int d\tau d\sigma \sqrt{\gamma} \gamma^{ab} \partial_a X^{\hat{\mu}} \partial_b X_{\hat{\mu}}. \quad (208)$$

The embedding functions can be split according to

$$X^{\hat{\mu}} = (X^{\hat{\mu}}, X^{D+2}), \quad \hat{\mu} = 0, 1, 2, \dots, D-1, D+1, \quad (209)$$

where  $X^{\hat{\mu}}$  occur in the action (12), whereas  $X^{D+2}$  is due to the additional dimension. Then we have

$$\partial_a X^{\hat{\mu}} \partial_b X_{\hat{\mu}} = \partial_a X^{\hat{\mu}} \partial_b X_{\hat{\mu}} + \partial_a X^{D+2} \partial_b X_{D+2}. \quad (210)$$

Inserting (210) into the action (208), we obtain

$$I = \frac{T}{2} \int d\tau d\sigma \sqrt{\gamma} \gamma^{ab} (\partial_a X^{\hat{\mu}} \partial_b X_{\hat{\mu}} + \partial_a X^{D+2} \partial_b X_{D+2}). \quad (211)$$

Variation of the action (211) with respect to  $\gamma_{ab}$  gives

$$\gamma_{ab} = f_{ab} \equiv \partial_a X^{\hat{\mu}} \partial_b X_{\hat{\mu}} = \partial_a X^{\hat{\mu}} \partial_b X_{\hat{\mu}} + \partial_a X^{D+2} \partial_b X_{D+2}. \quad (212)$$

Let us assume the following dependence<sup>8</sup> of the extra variable  $X^{D+2}$  on  $\xi^a = (\tau, \sigma)$ :

$$X^{D+2}(\tau, \sigma) = K^2 \tau, \quad (213)$$

so that

$$\partial_a X^{\hat{\mu}} \partial_b X_{\hat{\mu}} = \partial_a X^{\hat{\mu}} \partial_b X_{\hat{\mu}} + K^2. \quad (214)$$

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<sup>8</sup>Because of the reparametrization invariance, we are free to choose two of the  $D+2$  functions  $X^{\hat{\mu}}(\tau, \sigma)$ . We have already chosen  $X^{D+1} = \sigma$ . Now we make a choice for  $X^{D+2}$ .

Then we have

$$f_{11} = \partial_1 X^{\hat{\mu}} \partial_1 X_{\hat{\mu}} + K^2, \quad f_{12} = \partial_1 X^{\hat{\mu}} \partial_2 X_{\hat{\mu}}, \quad f_{22} = \partial_2 X^{\hat{\mu}} \partial_2 X_{\hat{\mu}} \quad (215)$$

which means that only  $f_{11}$  is modified by the presence of the additional,  $(D+2)^{\text{th}}$ , dimension. Instead of Eq. (7), we have

$$f_{ab} = \begin{pmatrix} \dot{x}^2 + K^2 + 2\dot{x}y k \sigma, & \dot{x}y k \\ \dot{x}y k, & k^2 y^2 + \epsilon \end{pmatrix}, \quad (216)$$

In the action (13),  $\dot{x}^2$  should be replaced by  $\dot{x}^2 + K^2$ . Such a modified action (13) gives the same equations of motion (19),(20),(21). Only Eq. (18), which contains  $\dot{x}^2$ , is modified. In view of (215), the extra term in the action (211) is then

$$I_1 = \frac{T}{2} \int d\tau d\sigma \sqrt{\gamma} \gamma^{11} K^2 = \frac{LT}{2} \int d\tau \frac{K^2}{e} + \mathcal{O}(k^2 L^2). \quad (217)$$

It contributes only to the  $e$  equation of motion. By repeating the calculations of Sec. 3 with inclusion of the extra term (217), we arrive at the following modified 1<sup>st</sup> class constraint:

$$\phi_1 = -\frac{m}{2} \Gamma^\mu \Gamma_\mu - \frac{\Pi^\mu \Pi_\mu}{4\mu} + \frac{1}{2m} (p^2 - m^2) - \frac{Q}{2}, \quad (218)$$

where

$$Q = \frac{mK^2}{e^2}. \quad (219)$$

Upon quantization, we have

$$\left[ -\frac{m}{2} \Gamma^\mu \Gamma_\mu - \frac{\Pi^\mu \Pi_\mu}{4\mu} + \frac{1}{2m} (p^2 - m^2) - \frac{Q}{2} \right] |\psi\rangle = 0, \quad (220)$$

where  $\Gamma^\mu$ ,  $\Pi^\mu$  are the operators, satisfying (157),(158), whereas  $p_\mu$  is now the momentum operator that commutes with  $\Gamma^\mu$ ,  $\Pi^\mu$  and can be represented as  $p_\mu = -i\partial/\partial x^\mu$ . By using Eqs. (204), the equation of state (220) becomes

$$\left[ -\frac{(D-1)m^2}{8\mu(m^2 - p^2)} + \frac{1}{2m} (p^2 - m^2) - \frac{Q}{2} \right] |\psi\rangle = 0. \quad (221)$$

Multiplication from the left by  $2m(m^2 - p^2)$  gives

$$\left[ -\frac{(D-1)m^3}{4\mu} - (m^2 - p^2)^2 - m(m^2 - p^2)Q \right] |\psi\rangle = 0. \quad (222)$$

This is the 4th order equation in the derivatives with respect to spacetime coordinates  $x^\mu$ ,  $\mu = 0, 1, 2, 3$ . Additionally, we also have the relation (205), and consequently the equation of state (222) becomes

$$\left[ -\frac{(D-3)m^3}{4\mu} - m\sqrt{\frac{m}{2\mu}}Q \right] |\psi\rangle = 0. \quad (223)$$

from which we obtain

$$Q = \frac{D-3}{2} \sqrt{\frac{m}{2\mu}} = \frac{(D-3)\omega}{2}. \quad (224)$$

Then the term  $Q/2$  that comes from the  $(D+2)^{\text{th}}$  time-like dimension cancels the residual term (207), occurring in Eq. (202). The same cancelation happens if instead of introducing the extra variable  $X^{D+2}$ , satisfying Eq. (213), we remain with  $D+1$  variables, such that one of them satisfies the analogous equation. Namely, if  $D$  is any dimension, then, of course, introducing an extra time-like dimension is equivalent to assuming that three, and not only two of the  $D+1$  dimensions are time-like, so that, for instance,  $X^{D-1} = K^2\tau$ , and  $\hat{\mu} = 0, 1, 2, \dots, D-2, D+1$ .

Because Eq. (223) is of such a simple form with only the  $c$ -numbers within the bracket, the equation that determines  $|\psi\rangle$  is in fact (205). This is the Klein-Gordon equation with the squared mass  $m^2 - m\sqrt{m/2\mu}$ . But because our physical states describe spin 1/2 particles, they must satisfy, not only the Klein-Gordon, but also the Dirac equation

$$(i\gamma^\mu \partial_\mu - m_e)|\psi\rangle, \quad m_e = \left(m^2 - m\sqrt{\frac{m}{2\mu}}\right)^{1/2}, \quad (225)$$

that is the “square root” of of Eq. (205). The above equation (225), and consequently its “square” (205), is a condition that has to be imposed in order to obtain the matching between the two different expressions, (168) and (169) for the spin operator. With (225), the circle is closed, and the representation of operators  $\Gamma^\mu$ ,  $\Pi^\mu$  in terms of the Clifford numbers according to Eqs. (164), (165), (174), (175), is consistent with the commutations relations (149)–(155), the definition of spin operator (156), and also with the equation of state (220).

## 5 Conclusion

We have investigated an open string living in a target space with an extra time-like dimension. As an approximation, more precisely, as a quenched description, we obtained a point particle with extrinsic curvature, which is responsible for the particle’s spin. The dynamics of such a system implies two first class and four second class constraints. We quantized this system by employing the Dirac brackets. We arrived at a system of operator equations that can be satisfied by Clifford numbers expressed in terms of the Dirac gammas and the generators of the Clifford algebra  $Cl(0, 2)$ . The spin operator of such quantized particle is expressed in terms of a superposition of the commutators  $[\gamma^\mu, \gamma^\nu]$ , the coefficients being the projectors onto the  $(D-1)$ -dimensional hypersurface, orthogonal to the particle’s momentum  $p_\mu$ ,  $\mu = 0, 1, 2, \dots, D-1$ . It turns out that the Pauli-Lubanski operator,  $S^\mu$ , calculated from the spin operator, is the same as that for the Dirac particle. The eigenvalues of  $S^\mu S_\mu$  are  $s(s+1) = 3/4$ . This means that our quantized system has spin  $s = 1/2$ .

One has to take into account also the first class constraints. Upon quantization, they become equations for physical states. But it turns out that with the Clifford algebra

representation of operators the condition  $\phi_1|\Psi\rangle = 0$  on physical states is not satisfied, unless the dimension of spacetime is  $D = 3$ . In order to render the equation of state consistent for  $D = 4$  or higher, one has to bring into the game one more extra time-like dimension, besides the two ones that are already present in our model. Altogether, if we take  $D = 4$ , we have thus three time like and three space like dimensions, i.e., an ultrahyperbolic space with neutral signature. Despite the fact that such spaces for certain well known reasons are considered as problematic for physics, there are works in the literature [34]–[40] which reveal just the contrary.

A remarkable feature of the model presented in this paper is that we started from a bosonic string, and then described the motion of its end at  $\sigma = 0$  as a point particle with extrinsic curvature (a variant of the so called ‘rigid particle’). Upon quantization, we obtained a spin 1/2 system. This is a consequence of another remarkable feature, namely that the algebra of the Dirac brackets becomes upon quantization the algebra of commutation relations between the operators that can be represented in terms of the gamma matrices. The spin operator, defined as the commutator of Clifford numbers, matches the spin operator, defined in terms of the velocity operator and its conjugate momentum, if the states satisfy the Dirac equation. That extended objects, even if apparently “bosonic”, can contain spinors, and thus spin one half states, is a result implied more or less explicitly in many works [31, 43]. Those approaches are based on the fact that the extended objects can be sampled by the center of mass coordinates, and the higher grade coordinates, describing the area, 3-volume, 4-volume,..., associated with the object. Such a description can be cast into an elegant form by means of Clifford algebras [31], and leads to the concept of Clifford space. It is well known since Cartan [44]–[46] that spinors are particular Clifford numbers. The fact that spinors are embedded in Clifford algebras has been explored in the literature in various contexts [47]. In the present paper we also sampled an extended object, but not in terms of Clifford numbers, but in terms of the variables  $x^\mu(\tau)$  that describe the motion of one of the string ends on a surface  $V_{D-1}$ , and the variables  $y(\tau)$  that, up to the first order in the expansion (5), describe the string’s extension into the direction orthogonal to  $V_{D-1}$ . Upon quantization, we arrived at the spin one half states. This is in agreement with what comes out from the Clifford algebra based approach to the extended objects. Because our model of the type 2a rigid particle and its quantization involves Clifford numbers, and because it is closely related to strings, it provides an additional test of the validity of the Clifford algebra description of strings. Moreover, we have confirmed and further elaborated an observation by some researchers [24]–[27] that the classical systems with spin resemble very closely the Dirac particle. There is a number of other works that consider various approaches to the classical particles with spin and their relation to the Dirac equation [48].

The connection of the type 2a rigid particle with strings, Clifford algebras and Clifford spaces [31, 43, 40], [49] which are very promising in our attempts to construct a unified theory of fundamental particles and interactions, including quantum gravity, makes the results of this paper a further step on our road towards the unification of the so far separate

pieces of physics. In this paper we considered the string with a finite length  $L$ , expanded around  $\sigma = 0$ . As an approximation we obtained the action for the point particle with curvature, where the terms of the order  $k^2 L^2$  and higher, that would give finer modulation of the trajectory  $x^\mu(\tau)$ , have been neglected. An interesting feature of the latter action is that it possess non trivial limit  $L \rightarrow 0$ , i.e.,  $m = LT \rightarrow 0$ , such that the term with curvature is still present. This means that if the string length  $L$  is decreasing, the string is becoming more and more like a rigid particle. In the limit in which the string is infinitely short, it behaves as the  $m = 0$  rigid particle, considered by McKeon [19]. We have touched this topics only in Footnote 2, whereas a detailed investigation of the dynamics of such a particle has been considered in Ref. [50].

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## Appendix: The Laplace equation for a time-like string and its boundary conditions

Let us investigate in more detail the situation of a string whose ends are attached to two  $Dp$ -branes that sweep the worldvolumes  $V_D$  and  $V'_D$ , as illustrated in Fig. 1.

In the case of a usual, *space-like string* ( $\epsilon = -1$ ), sweeping a worldsheet with signature  $(+ -)$ , the string's embedding functions  $X^\mu(\tau, \sigma)$  satisfy in the conformal gauge the Helmholtz equations of motion, subjected to a combination of Von Neumann and Dirichlet boundary conditions, so that the string ends in general move on a  $D$ -brane which in the case considered in Fig. 1, is  $Dp$ -brane, with  $p = D - 1$ , sweeping the worldvolume  $V_{D1}$ . The  $(D + 1)^{\text{th}}$  dimension, along which the string extends, is space-like.

The situation is different in the case of a *time-like string* ( $\epsilon = 1$ ), which corresponds to the worldsheet signature  $(+ +)$  and time-like  $X^{D+1}$ , then in the conformal gauge we have the following equations of motion and the constraints:

$$\ddot{X}^{\hat{\mu}} + X''^{\hat{\mu}} = 0, \quad \dot{X}^{\hat{\mu}} \dot{X}_{\hat{\mu}} - X'^{\hat{\mu}} X'_{\hat{\mu}} = 0, \quad \dot{X}^{\hat{\mu}} X'_{\hat{\mu}} = 0. \quad (226)$$

Each embedding function thus satisfies the Laplace equation, subjected to the above constraints. A general solution is

$$X^\mu = C^\mu \tau + \sum_n (a_n^\mu \cos \omega_n \tau + b_n^\mu \sin \omega_n \tau) (A_n e^{k_n \sigma} + B_n e^{-k_n \sigma})$$

$$X^{D+1} = \sigma, \quad \sigma \in [0, L], \quad (227)$$

where

$$\omega_n^2 - k_n^2 = 0, \quad a_n^2 = b_n^2, \quad C_\mu a_n^\mu = C_\mu b_n^\mu = a_n^\mu b_{n\mu} = 0, \quad C^2 = 1. \quad (228)$$

A particular solution is determined if the values of the functions  $X^{\hat{\mu}}(\tau, \sigma)$  on the boundary are given (Dirichlet boundary conditions):

$$X^{\mu}(\tau, 0) = x^{\mu}(\tau) , \quad X^{D+1}(\tau, 0) = 0, \quad (229)$$

$$X^{\mu}(\tau, L) = g^{\mu}(\tau) , \quad X^{D+1}(\tau, L) = L, \quad (230)$$

$$X^{\mu}(\tau_1, \sigma) = F^{\mu}(\sigma), \quad X^{D+1}(\tau_1, \sigma) = \sigma, \quad (231)$$

$$X^{\mu}(\tau_2, \sigma) = G^{\mu}(\sigma) , \quad X^{D+1}(\tau_2, \sigma) = \sigma. \quad (232)$$

Now a question arises as to how an observer could manage such boundary conditions on the branes  $V_D$  and  $V'_D$ . Mathematically, of course we can say that the string's worldsheet  $V_2$  satisfies such boundary conditions and the equations of motion (226). But an observer, like us, cannot see the data in the entire space  $M_{D+1}$ . According to our brane world scenario, the observers live in one of the two branes, or both. Let us assume that the observers live in the brane  $V_D$  at  $\sigma = 0$ , and that they can only observe and determine the data in  $V_D$ . An observer in  $V_D$  can thus only trace the functions  $X^{\mu}(\tau, 0) = x^{\mu}(\tau)$ , but not the entire string. From the point of view of such an observer, functions  $x^{\mu}(\tau)$  could be dynamical variables, if they satisfied certain equations of motion. We have seen that, starting from the action (1), we can indeed derive dynamical equations of motion for  $x^{\mu}(\tau)$ . This can be done if we expand the worldsheet embedding functions  $X^{\mu}(\tau, \sigma)$  into a Taylor series around the point  $\sigma = 0$ , according to Eq. (5).

In Eqs. (226)–(232) we worked in a particular gauge, namely the conformal gauge, whereas in the derivation from Eq. (1) to Eq. (17), we did not specify a gauge. A gauge is determined by a choice of the Lagrange multipliers  $e$  and  $f$ . If  $e = 1$  and  $f = 0$ , then from Eqs. (14)–(16) we have  $\gamma^{11} = \frac{1}{\sqrt{\gamma}}$ ,  $\gamma^{22} = \frac{1}{\sqrt{\gamma}}$ ,  $\gamma^{12} = 0$ , which means that the metric is conformal. In conformal gauge, the equations of motion  $\dot{p}_{\mu} = 0$  with  $p_{\mu}$  given in Eq. (42) become

$$\ddot{x}^{\mu} + \omega^2 \ddot{x}^{\mu} = 0 , \quad \omega = \sqrt{\frac{m}{2\mu}} = \frac{2}{L}. \quad (233)$$

A general solution is

$$x^{\mu}(\tau) = C^{\mu}\tau + a^{\mu}\cos\omega\tau + b^{\mu}\sin\omega\tau. \quad (234)$$

Using (23), we find

$$y^{\mu}(\tau) = \frac{L}{2k} \ddot{x}^{\mu} = \frac{\omega^2 L}{2k} (a^{\mu}\cos\omega\tau + b^{\mu}\sin\omega\tau). \quad (235)$$

Since  $x(\tau) = X^{\mu}(\tau, 0)$  and  $y^{\mu}(\tau) = \frac{1}{k} X'^{\mu}(\tau, 0)$ , we see that Eqs. (234) and (235) provide boundary conditions at  $\sigma = 0$  for the general solution (227). From Eq. (227) we have

$$X^{\mu}(\tau, 0) = C^{\mu}\tau + \sum_n (a_n^{\mu}\cos\omega_n\tau + b_n^{\mu}\sin\omega_n\tau)(A_n + B_n), \quad (236)$$

$$X'^{\mu}(\tau, 0) = \sum_n (a_n^{\mu}\cos\omega_n\tau + b_n^{\mu}\sin\omega_n\tau)k_n(A_n - B_n). \quad (237)$$

This coincides with Eqs. (234) and (235), if  $k_1 = k = \omega = 2/L$ ,  $A_1 = 1$ ,  $a_1^\mu = a^\mu$ ,  $b_1^\mu = b^\mu$  and  $A_n = 0$ ,  $a_n^\mu = 0$ ,  $b_n^\mu = 0$  for  $n \neq 1$ . The general solution in the presence of the boundary conditions (234),(235) thus becomes

$$\begin{aligned} X^\mu &= C^\mu \tau + (a^\mu \cos \omega \tau + b^\mu \sin \omega \tau) e^{k\sigma}, \\ X^{D+1} &= \sigma, \quad \sigma \in [0, L]. \end{aligned} \quad (238)$$

It is illustrative to plot the solution (238). For this aim let us take  $C^\mu = (1, 0, 0, 0)$ ,  $a^\mu = (0, 1, 0, 0)$ ,  $b^\mu = (0, 0, 1, 0)$ ,  $\omega = 1$ . We obtain a helix like solution given in Figs. 2 and 3 for two different values of  $L$ .

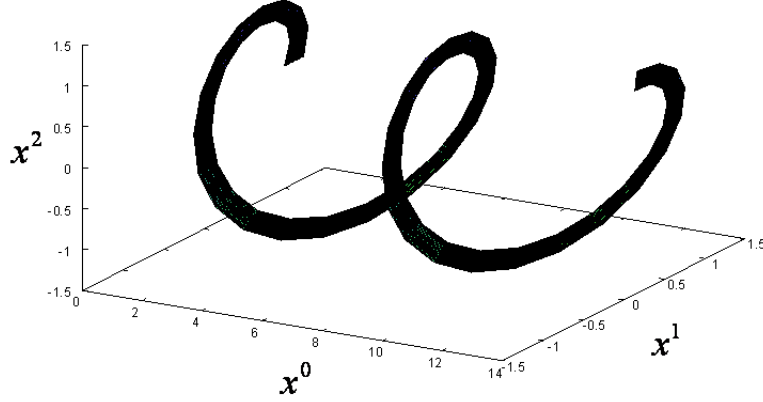


Figure 2: The projection of the 2-dimensional worldsheet, given by Eq. (238), onto the subspace, described by coordinates  $(x^0, x^1, x^2)$ , if the string is short. We see that the boundary curve for such a short string is a good approximation to the string worldsheet.

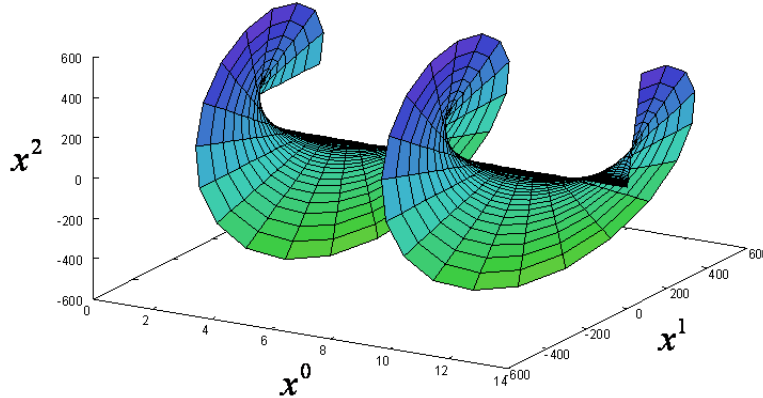


Figure 3: The projection of the 2-dimensional worldsheet, given by Eq. (238), onto the  $(x^0, x^1, x^2)$ -subspace, if the string is long.



By solving the equations of motion (233),(235) for  $x^\mu(\tau)$  and  $y^\mu(\tau)$ , we have thus found boundary conditions for the string that satisfies the equations of motion (226). A boundary at  $\sigma = 0$  of our time-like string is the point particle with extrinsic curvature. Of course, this is not the only possible boundary. If we took more terms in the expansion (5), then we would obtain the equations of motion for more terms  $y_i^\mu(\tau)$ , whose solution would be a more complicated curve than the boundary  $\sigma = 0$  in Eq. (238), and a more complicated worldsheet than in Fig. 2 or Fig. 3.

Finally, let me comment on the boundary conditions that an open string, either space-like or time-like has to satisfy. First it was believed that an open (space-like) string must satisfy the Von Neumann boundary conditions. Later it was found that it can satisfy the Dirichlet boundary conditions, as well as a combinations of those two types of conditions, which led to the discovery of  $D$ -branes.

This suggests a thesis that any possible solution of the string equations of motion can in principle be realized in Nature. One should not impose certain boundary conditions and consider only those solutions that satisfy them. Every solution corresponds to a possible physical situation related to how the string ends are coupled (or not coupled) to other physical objects (e.g., strings and branes). In a quantized theory, all those possibilities have to be taken into account in the wave functional. In fact, this is the essence of quantum theory. While in the classical theory of a point particle there are differential equations of motion that have a class of possible solutions, and one has to choose one particular solution by imposing some initial conditions, in the quantum theory all those possibilities are contained in the wave function. Analogous must hold for extended objects, such as strings that satisfy partial differential equations. They allow for a class of possible solutions, and one has to specify initial and boundary conditions in order to obtain one particular solution. Usually, in the quantized string theory all possible solutions corresponding to all possible initial conditions are taken into account, but there is still a restriction due to choice of boundary conditions. In a complete quantization, one should take into account all possible boundary conditions as well, and thus the total class of possible solutions. This does not mean that a “partial” quantization of the string theory is not correct, it only does not take into account the whole story. An important step forward into this direction was the the discovery of  $D$ -branes as dynamical objects.

In the case of a time-like string that satisfies the Laplace equation (226), the situation is more straightforward, because here the Laplace equation is not, like in electrostatics, a static case of a Helmholtz equation, and therefore there is no need to care about the momentum transfer across the string ends. All directions along the string worldsheet are time-like. However, a more than one-dimensional time is not consistent with the dynamics as we experience it. In order to obtain a one dimensional time, one has to intersect the two-time worldsheet  $V_2$  with a space-like surface  $\Sigma \subset V_{D+1}$  and consider the dynamics on  $\Sigma$ . It must be a consistent dynamics, with conserved energy-momentum. In our scenario, the space-like surface  $\Sigma$  is the brane  $V_{D-1}$  that does not intersect, but touches  $V_2$  at the

string end and sweeps the worldvolume  $V_D$ . The dynamical equations of the motion of the string end at  $\sigma = 0$ , considered in this paper, are the equations for a point particle with extrinsic curvature, with the conserved momentum  $p_\mu$ . They were obtained by expanding the string embedding functions  $X^\mu(\tau, \sigma)$  in terms of the parameter  $\sigma$  up to the first order. Inclusion of higher orders in the expansion (5) would modify the point particle action so that besides the extrinsic curvature it would contain torsion and higher curvatures as well. In the corresponding equations of motion not only fourth, but also higher derivatives would enter the game. A solution of such equations of motion would also be consistent with the boundary of a time-like Nambu-Goto string. Regardless of how many terms we take in the expansion of  $X^\mu(\tau, \sigma)$ , we always obtain an exact boundary of the string, satisfying certain higher derivative equations of motion. It is reasonable to anticipate that for a given dimension  $D$  the expansion (5) above certain order brings nothing new to the equations of motion of the boundary point. Namely, the expansion to the first order gives the point-particle with extrinsic curvature, whereas the expansion to higher orders presumably gives the point particle action with higher curvatures. Because according to the Frenet formula in  $D$ -dimension there are  $D - 1$  generalized curvatures associated with a curve, this imposes the limit to how many curvature terms can be in the action for a point particle whose worldline traces the motion of the string end. The reasoning along such lines thus leads to the point particle with  $D - 1$  generalized curvatures, where the point particle with extrinsic curvature, considered in this paper, is just a special case.

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